October 27

Not all flows can be defined globally.

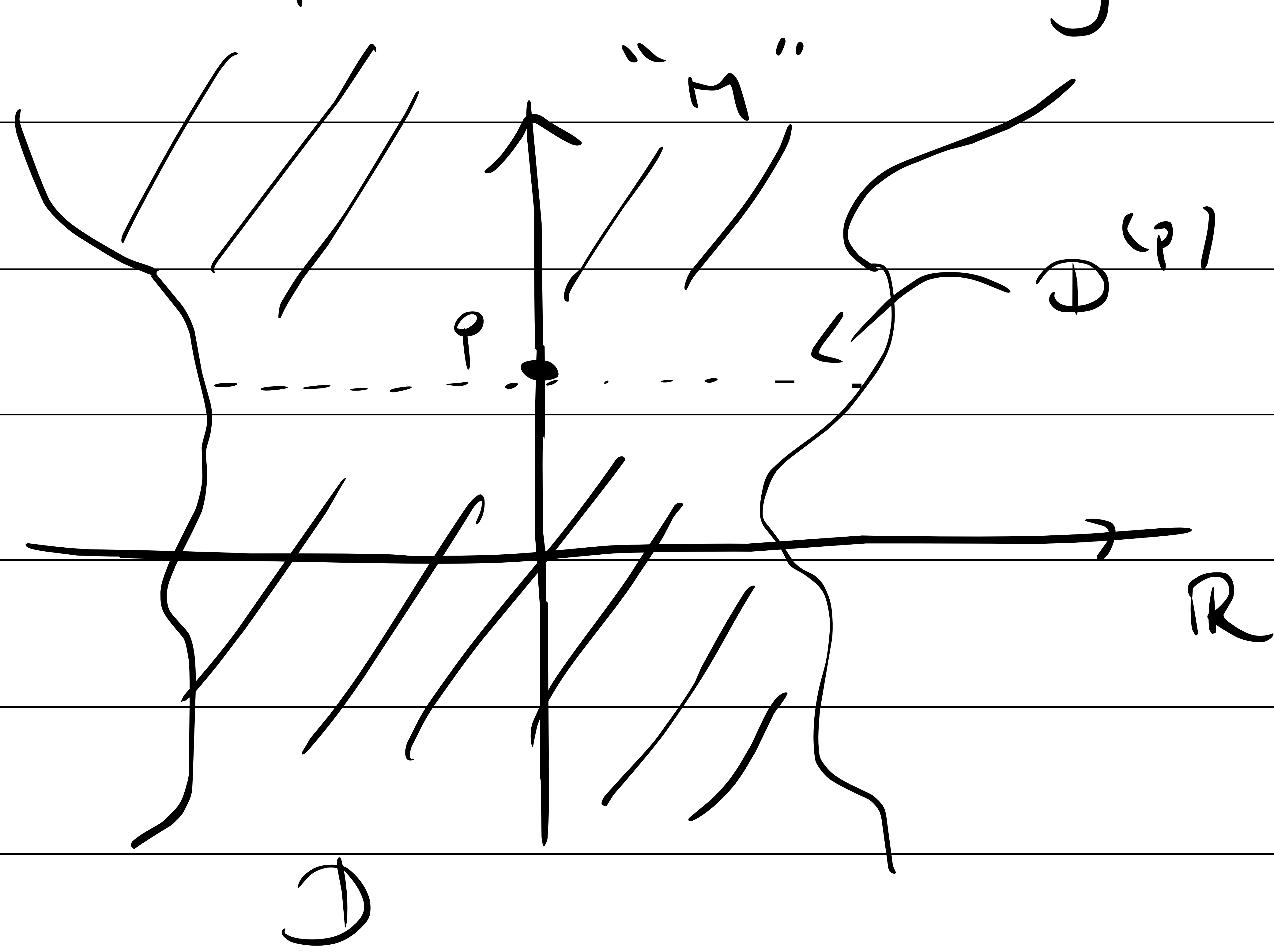
A **flow domain** is an open

$$\mathcal{D} \subset \mathbb{R} \times M$$

s.t. $\forall\ p \in M$ the open set

$$\mathcal{D}^{(p)} = \left\{ t \in \mathbb{R} \;\middle|\; (t, p) \in \mathcal{D} \right\}$$

is an open set containing $0 \in \mathbb{R}$.



A <u>local flow</u> is a smooth map

$$\theta : \mathcal{D} \longrightarrow M$$

s.t. for all $p \in M$:

1) $\theta(0, p) = p$.

2) For all $s \in \mathcal{D}^{(p)}$ and $t \in \mathcal{D}^{\theta(s,p)}$ s.t. $s+t \in \mathcal{D}^{(p)}$ we have

$$\theta(t, \theta(s,p)) = \theta(t+s, p).$$

As before, we call the v.f. V defined by

$$V_p = \theta_{\cdot}(p)'(0)$$

the infinitesimal generator of θ.

- A <u>maximal integral curve</u> is one that cannot be extended to an integral curve on any bigger open interval. Likewise, a maximal flow is one that cannot be extended to a bigger flow domain.

<u>Thm</u>: Let $V \in \mathrm{Vect}(M)$. There is a unique maximal local flow

$$\theta : \mathcal{D} \to M$$

whose infinitesimal generator is V.

In other words, near each $p \in M$ a vector field determines a flow in some neighborhood of p.

Next, we move on to a generalization of the concept of a "directional derivative".

Lie Derivatives

Given $v \in T_pM$ and $f \in C^\infty(M)$, we have the quantity

$$vf = D_v f\big|_p \in \mathbb{R}.$$

This is a generalization of the directional derivative of f at p.

What about the directional derivative of a <u>vector field</u>?

In $\mathbb{R}^n$ we can do this. If X is a v.f. on $\mathbb{R}^n$ then:

$$D_v X\Big|_p = \frac{d}{dt}\Big|_{t=0} X_{p+tv}$$

$$= \lim_{t\to 0} \frac{X_{p+tv} - X_p}{t}.$$

Writing $X = \sum_i X^i \frac{\partial}{\partial x^i}$, $X^i \in C^\infty(\mathbb{R}^n)$,

then this is

$$D_v X\Big|_p = \sum_i \left(D_v X^i\Big|_p\right) \frac{\partial}{\partial x^i}\Big|_p.$$

This uses the structure of $\mathbb{R}^n$ as a vector space! Indeed:

$$p \rightsquigarrow p + tv$$

is a maneuver that only makes sense in $\mathbb{R}^n$.

Attempt at a resolution: replace the linear curve $\gamma(t) = p + tv$ by any curve in M s.t.

$$\gamma(0) = p, \quad \gamma'(0) = v.$$

But still we have a problem:

$$X_{\gamma(t)} - X_{\gamma(0)} \quad \text{does not make sense!}$$

Indeed, in $\mathbb{R}^n$ we have used the canonical isomorphism $T_{\gamma(0)}\mathbb{R}^n \cong T_{\gamma(t)}\mathbb{R}^n$.

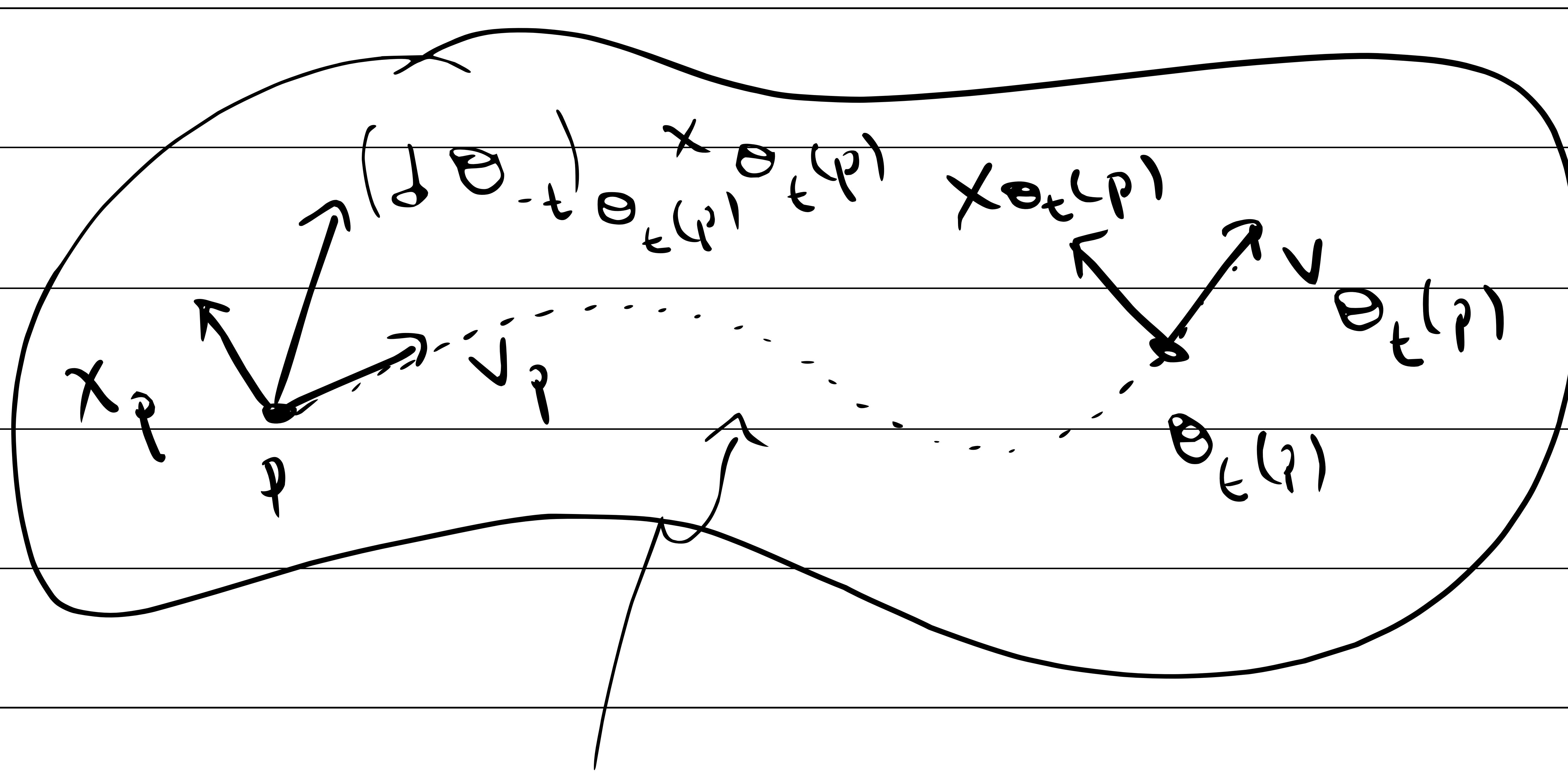
We fix this by first changing $v \in T_pM$, to a v.f. $V \in \mathrm{Vect}(M)$.

We then use the flow of V to "move" $X_{\gamma(0)}$ and $X_{\gamma(t)}$ to line in the same space. This is defined in a nbd of p.

Let θ_t be the local flow corresponding to V. Define the Lie derivative of X with respect to V as

$$(L_V X)_p = \frac{d}{dt}\Big|_{t=0} d(\theta_{-t})_{\theta_t(p)}\left(X_{\theta_t(p)}\right)$$

$$= \lim_{t \to 0} \frac{1}{t}\left(d(\theta_{-t})_{\theta_t(p)}\left(X_{\theta_t(p)}\right) - X_p \right).$$



$\theta_{\cdot}(p)$ is the curve.

Lem: $p \mapsto (\mathcal{L}_V X)_p$ defines a smooth vector field on M denoted $\mathcal{L}_V X$.

Pf: (U, ϕ) chart containing p. Let J_0 be open interval containing 0 and $U_0 \subset U$ be open s.t.

$$\theta(J_0 \times U_0) \subset U.$$

Have component fns $\left(\theta^1(t,x), \ldots, \theta^n(t,x)\right)$, and $(d\theta_{-t})_{\theta_t(x)}$ is represented by the matrix

$$\left(\frac{\partial \theta^i}{\partial x^j}\left(-t, \theta_t(x)\right) \right).$$

So

$$(d\theta_{-t})_{\theta_t(x)}\left(X_{\theta_t(x)}\right)$$

$$= \frac{\partial\theta^i}{\partial x^j}\left(-t,\theta_t(x)\right) X^j\left(\theta(t,x)\right)\frac{\partial}{\partial x^i}\Big|_x .$$

Because θ^i, X^j are smooth it follows that this is smooth. $\blacksquare$

October 30

Thm: $L_V X = [V, X]$ for any v.f.'s V, X.

Pf: Let $R(V) = \{p \in M \mid V_p \neq 0\} \subset M$.

This is open since V is cts. We show

$$(L_V X)_p = [X, Y]_p .$$