September 14

Last time we defined what it means for a f^n

$$f : M \longrightarrow \mathbb{R}$$

to be smooth.

Now suppose M, N are smooth manifolds, of dim m, n respectively.

<u>Dfn</u>: A map

$$F : M \longrightarrow N$$

is <u>smooth</u> if for every $p \in M$ there exists smooth charts (U, ϕ) containing p and (V, ψ) containing $F(p)$ s.t. $F(U) \subset V$ and:

$$\begin{array}{ccccccc} \phi(U) & \xrightarrow{\phi^{-1}} & U & \xrightarrow{F} & V & \xrightarrow{\psi} & \psi(V) \\ \cap & & & & & & \cap \\ \mathbb{R}^m & & & & & & \mathbb{R}^n \end{array}$$

is smooth.

<u>Rmk</u>: • Every smooth map is continuous.

- Smoothness is local. That is:

  - if every $p \in M$ has a nbd s.t.

$$F|_U : U \longrightarrow N$$

is smooth, then F is smooth.

  - conversely if F is smooth and $U \subset M$ is open, then $F|_U$ is smooth.

Expanding on this last rmk. Sps $\{U_\alpha\}$ is a cover of M (not necessarily by coordinate charts). Suppose for each α we have a map

$$F_\alpha : U_\alpha \longrightarrow N$$

which is smooth. Assume that

$$F_\alpha\Big|_{U_\alpha \cap U_\beta} = F_\beta\Big|_{U_\alpha \cap U_\beta}$$

Then there exists a unique smooth map

$$F : M \longrightarrow N$$

s.t. $F\big|_{U_\alpha} = F_\alpha$. $\boxed{\#}$

<u>Ex</u>:

- $\mathbb{1} : M \to M$ is smooth

- If $q \in N$ then the constant map

$$F_q : M \longrightarrow N \quad \text{is smooth.}$$
$$p \longmapsto q$$

- If $U \subset M$ is open, then

$$i_U : U \longrightarrow M \quad \text{is smooth.}$$
$$\begin{array}{ccc} \cup & & \cup \\ x & \longmapsto & x \end{array}$$

- The composition of smooth fns is smooth.

More examples:

- Let $e : \mathbb{R} \longrightarrow S^1$ be

$$e(t) = e^{2\pi i t}.$$

Then e is smooth.

[Let $U \subset S^1$ be open and sps $U \neq S^1$. Then there is a smooth chart (U, θ) where

$$\theta : U \longrightarrow \mathbb{R}$$

is s.t. $e^{i\theta(z)} = z$. In such a coordinate chart we have that

$$\theta \circ e(t) = 2\pi t + c$$

where c is some constant.]

• The map $i: S^n \longrightarrow \mathbb{R}^{n+1}$

$$(x^1, \dots, x^{n+1}) \longmapsto (x^1 \dots x^{n+1})$$

is smooth.

[In the coordinate chart $\phi_i^{\pm}$ we have

$$i \circ (\phi_i^{\pm})^{-1}(u^1, \dots, u^n) =$$

$$(u^1, \dots, u^{i-1}, \pm\sqrt{1-|u|^2}, u^i, \dots, u^n) .$$
] //

__Dfn__: A __diffeomorphism__ is a smooth map

$$F: M \longrightarrow N$$

which is bijective and F^{-1} is smooth.

__Prop__: Let $B^n = \{x \in \mathbb{R}^n \mid |x| < 1\}$, define

$$F: B^n \to \mathbb{R}^n, \quad F(x) = \frac{x}{\sqrt{1-|x|^2}} .$$

Then F is a diffeomorphism.

Pf: It is clear F is smooth. The inverse is

$$F^{-1}(y) = \frac{y}{\sqrt{1+|y|^2}},$$

which is also smooth. ▨

- If (U, ϕ) is any smooth chart, then

$$\phi : U \longrightarrow \phi(U)$$

is a diffeomorphism.

Thm: 1) F, G diffeos $\Rightarrow$ $F \circ G$ diffeo.

2) diffeomorphism is an equivalence relation on the class of smooth manifolds.

3) If $\exists$ a diffeo $F: M \to N$, then
$$\dim M = \dim N.$$

Pf: 1) follows from chain rule.

2) Easy.

3) Let $p \in M$ and let (U, ϕ), (V, ψ) be charts containing p, $F(p)$ respectively. Then
$$\psi \circ F \circ \phi^{-1}$$
is a diffeomorphism from an open subset in $\mathbb{R}^m$ to an open subset in $\mathbb{R}^n$. By our result from two lectures ago we see that $n = m$. $\blacksquare$

We say M and N are <u>diffeomorphic</u> if $\exists$ diffeomorphism $F: M \to N$. This is an equivalence relation on smooth manifolds.

Prop: Let $\tilde{\mathbb{R}}$ be the smooth str on the real line defined by the chart

$$(\mathbb{R}, \psi = x^3)$$

Then $\tilde{\mathbb{R}}$ is diffeomorphic to the standard smooth str on $\mathbb{R}$.

Pf: The diffeomorphism

$$F : \mathbb{R} \longrightarrow \tilde{\mathbb{R}}$$

is

$$F(x) = x^{1/3}.$$

The coordinate rep is

$$\psi \circ F = \mathbb{1}, \quad \text{so smooth}.$$

The coordinate rep for F^{-1} is

$$F^{-1} \circ \psi^{-1} = \mathbb{1}, \quad \text{also smooth.} \qquad \boxtimes$$

In fact <u>all</u> smooth str's on $\mathbb{R}$ are diffeomorphic.

<u>Rmk</u>: The only Euclidean space which admits "exotic" smooth structures up to diffeomorphism is $\mathbb{R}^4$.

First constructed by Donaldson, Freedman ...