September 25

The "nicest" smooth maps are diffeomorphisms.

Here is a road map to the next few lectures:

diffeomorphisms

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(smooth) embeddings $\qquad$ local diffeomorphism $\qquad$ surjective submersions

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immersions $\qquad$ submersions

$\cap$ $\qquad$ $\cap$

smooth maps of constant rank

$\cap$

smooth maps

Dfn: A smooth map $F: M \to N$ is of <u>constant rank</u> if $dF_p : T_pM \to T_{F(p)}N$ has constant rank for all $p \in M$.

A map $\overset{F}{}$ of constant rk is:

- An <u>immersion</u> if dF_p is injective for all p.

- A <u>submersion</u> if dF_p is surjective for all p.

<u>Prop</u>: $F: M \to N$ smooth. If $p \in M$ is s.t. dF_p is injective/surjective then $\exists$ a nbd $\mathcal{U} \ni p$ s.t

$$F\big|_{\mathcal{U}} : \mathcal{U} \longrightarrow N$$

is immersion/submersion.

<u>Pf</u>: If we choose a chart $(\mathcal{U}, \phi)$ of M and compatible chart (V, ψ) of N, then dF_p is represented by the Jacobian matrix

of partial derivatives. This gives us a map

$$dF_p : U \longrightarrow Mat_{n \times m}$$

<u>Lemma</u>: Let

$$W = \left\{ \begin{matrix} \text{matrices w/} \\ \text{maximal rank} \end{matrix} \right\} \subset Mat_{n \times m}.$$

Then W is <u>open</u>.

<u>Pf</u>: wlog sps $m < n$. Then $A \in W \Rightarrow$

$\exists$ submatrix of size $m \times m$ (obtained by deleting some rows + columns) which is invertible.

So

$$f(A) \overset{\text{dfn}}{=} \sum_{\substack{B \in \text{submatrix} \\ \text{of size } m \times m \\ \text{of } A}} |\det(B)| > 0.$$

But $A \longmapsto f(A)$ is a cts map

$$f : M_{n\times m} \longrightarrow \mathbb{R}$$

$\Rightarrow$

$$W = f^{-1}(\mathbb{R}^{\times}) \quad \text{is open.} \quad \boxed{\text{///}}$$

Back to proof. Since

$$dF_p : U \longrightarrow M_{n\times m}$$

is cts (in fact smooth) we see that

$$\underset{\underset{p}{\Downarrow}}{(dF_p)^{-1}(W)} \underset{\text{open}}{\subset} U \subset M.$$

$\boxed{\text{///}}$

We move on to some examples involving rank, immersion/submersion.

Ex: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be

$$f(x, y) = xy + x^2.$$

Then $$df_{(x,y)} = \left(y + 2x \mid x \right)$$

For $(x, y) \neq (0,0)$ the rank of $df_{(x,y)}$ is 1, but $df_{(0,0)} = 0$. So f does not have constant rank.

Ex: Let $g : \mathbb{R}^2 \to \mathbb{R}$ be

$$g(x, y) = xy + x^2 - y^2 + y$$

Then

$$dg_{(x,y)} = \left(y + 2x \mid 2x - y + 1 \right).$$

Submersion

Inverse Function Thm

Locally:

$$\begin{array}{ll} U & \overset{\text{open}}{\subset} \mathbb{R}^n \\ \text{smooth } F \big\downarrow & \\ V & \subset_{\text{open}} \end{array}$$

<u>Thm</u>: If $\exists$ $a \in U$ s.t. $DF\big|_a$ is invertible, then $\exists$ open nbd $\tilde{U}$ w/ $a \in \tilde{U} \subset U$ s.t.

$$F\big|_{\tilde{U}} : \tilde{U} \to V$$

is a diffeo onto its image. Moreover

$$D\left(F\big|_{\tilde{U}}\right)^{-1}\Big|_{F(a)} = \left(D F\big|_{\tilde{U}}\Big|_a\right)^{-1}$$

<u>Pf</u>: single variable case: If

$$f : \mathbb{R} \to \mathbb{R}$$

has $f'(a) \neq 0$ for some a, then

by continuity of $f' : \mathbb{R} \to \mathbb{R}$ $\exists$ some nbd $\mathcal{U} \ni a$ s.t. $f'|_{\mathcal{U}} > 0$. Then the mean value theorem says that f is increasing on $\mathcal{U}$ $\Rightarrow$ $f|_{\mathcal{U}} : \mathcal{U} \to f(\mathcal{U})$ is smooth and bijective. Apply same reasoning to f^{-1}. ☒

We will not prove the general case.

Ex: $F : \mathbb{R}^2 \to \mathbb{R}^2$

$$F(x,y) = (y, x^2).$$

Then

$$DF|_{(x,y)} = \begin{pmatrix} 0 & 1 \\ 2x & 0 \end{pmatrix}$$

Not invertible at $(0,0)$. There exists no nbd $\mathcal{U}$ of $(0,0)$ s.t. $F|_{\mathcal{U}}$ is injective.

Ex: Let

$$\mathcal{U} = \left\{ (r,\theta) \;\middle|\; \begin{matrix} r > 0 \\ 0 < \theta < \pi \end{matrix} \right\} \subset \mathbb{R}^2$$

and define $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$F(r,\theta) = (r\cos\theta \,,\, r\sin\theta).$$

Have $DF\big|_{(r,\theta)} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$

invertible on $\mathcal{U}$, but not all of $\mathbb{R}^2$!

$$\left(F\big|_{\mathcal{U}}\right)^{-1}(x,y) = \left(\sqrt{x^2+y^2}\,,\ \arccos\frac{x}{r}\right).$$

We will use the inverse function theorem for manifolds.

Thm: $F: M \longrightarrow N$ is smooth. If $p \in M$ is s.t. dF_p is invertible then $\exists$ nbd $\tilde{\mathcal{U}}$ of p s.t.

$$F\big|_{\tilde{\mathcal{U}}} : \tilde{\mathcal{U}} \to F(\tilde{\mathcal{U}})$$

is a diffeomorphism. //

So: if F is s.t. dF_p is invertible for all $p \in M$ then it is a "local diffeomorphism".

Dfn: $F: M \longrightarrow N$ is a <u>local diffeomorphism</u> if for every $p \in M$ $\exists$ nbd $\mathcal{U}$ of p s.t.

$$F\big|_{\mathcal{U}} : \mathcal{U} \longrightarrow F(\mathcal{U})$$

is a diffeomorphism.