<u>September 27</u> | Last time we ended w/ the inverse fn thm. Now for manifolds:

<u>Thm</u>: $F : M \longrightarrow N$ is smooth. If $p \in M$ is s.t. dF_p is invertible then $\exists$ nbd $\tilde{U}$ of p s.t.

$$F\big|_{\tilde{U}} : \tilde{U} \to F(\tilde{U})$$

is a diffeomorphism. //

So: if F is s.t. dF_p is invertible for all $p \in M$ then it is a "local diffeomorphism".

<u>Dfn</u>: $F : M \longrightarrow N$ is a <u>local diffeomorphism</u> if for every $p \in M$ $\exists$ nbd U of p s.t.

$$F\big|_{U} : U \longrightarrow F(U)$$

is a diffeomorphism.

Dfn: A smooth map $F: M \to N$ is a **local diffeomorphism** if for every $p \in M$, $\exists$ nbd $U \ni p$ s.t. $F|_U : U \to F(U)$ is a diffeomorphism.

Prop: 1) F is local diffeo $\Leftrightarrow$ it is both an immersion and submersion.

2) If $\dim M = \dim N$ any submersion or immersion is a local diffeomorphism.

Pf: 1) Follows from inverse function thm.

2) dimension counting. $\boxtimes$

Ex: The map $e: \mathbb{R} \to S^1$
$$t \mapsto e^{it}$$
is a local diffeomorphism since

it is a submersion. Indeed, around every pt in S^1, there is a chart s.t.

$$e(t) = 2\pi t + c$$ ← some constant.

<u>Rank theorem</u>: The rank theorem provides a "standardized" local form of any map of constant rank. Sps $F : M \to N$, and $(U, \phi), (V, \psi)$ charts for M, N resp.

The coordinate repⁿ for F is

$$\begin{tikzcd} \phi(U) \arrow[r, "\phi^{-1}"] \arrow[rrr, bend right] & U \arrow[r, "F"] & V \arrow[r, "\psi"] & \psi(V) \end{tikzcd}$$

$\phi(U) \subset \mathbb{R}^m$

$$\hat{F} = \psi \circ F \circ \phi^{-1}.$$

<u>Thm</u>: If F is constant rank r. There exits $(\tilde{U}, \phi)$ of M and $(\tilde{V}, \psi)$ of N near every pt $p \in M$, $F(p) \in N$ such that

$$\hat{F}(x^1,\dots,x^r,x^{r+1},\dots,x^m) = (x^1,\dots,x^r,0,\dots,0).$$

in this coordinate rep, where $r \le \min\{m,n\}$.

In particular if F is submersion then

$$\hat{F}(x^1,\dots,x^m) = (x^1,\dots,x^n)$$

and if F is immersion then

$$\hat{F}(x^1,\dots,x^m) = (x^1,\dots,x^m,0,\dots,0).$$

Pf: We can replace M, N by open subsets $p = 0 \in U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ $\ni$ $F(p) = 0$.

By a linear change of coordinates, we can assume that

$$\left(\frac{\partial F^i}{\partial x^j}\right)_{i,j=1,\dots,r} \text{ is invertible.}$$

We use coordinates

$$(x,y) = (x^1, \dots, x^r, y^1, \dots, y^{m-r}) \text{ for } \mathbb{R}^m$$

$$(v,w) = (v^1, \dots, v^r, w^1, \dots, w^{n-r}) \text{ for } \mathbb{R}^n.$$

Doing this we write $F = (Q, R)$, i.e.:

$$F(x,y) = \big(Q(x,y), R(x,y) \big)$$

where $Q : U \longrightarrow \mathbb{R}^r$

$R : U \longrightarrow \mathbb{R}^{n-r}$

and $\left(\dfrac{\partial Q^i}{\partial x^j} \right)_{i,j=1,\dots,r}$ is invtble at 0.

Let $\phi : U \to \mathbb{R}^m$ be $\phi(x,y) = (Q(x,y), y)$.

$\rightsquigarrow$

$$D\phi\big|_{(0,0)} = \left(\begin{array}{c|c} \left.\dfrac{\partial Q^i}{\partial x^j}\right|_{(0,0)} & \left(\dfrac{\partial Q^i}{\partial y^j}\right)(0,0) \\ \hline 0 & \mathbb{1} \end{array} \right).$$

Continue next time...