MA 725 - DIFFERENTIAL GEOMETRY, I

HOMEWORK 1

Problem 1. Let V be an n-dimensional real vector space equipped with a symmetric, non-degenerate bilinear form $\langle -, - \rangle \colon V \times V \to \mathbf{R}$.

(1) Show that V admits a splitting $V = P \oplus N$ where $\langle -, - \rangle$ is positive definite on P and negative definite on N. Show that $p = \dim P$ is unique, that is, depends only on the form $\langle -, - \rangle$. The pair of integers $(\dim P, \dim N)$ is called the *signature* of $\langle -, - \rangle$.
(2) Show that there exists a basis $\{e_i\}$ for V such that $\langle e_i, e_j \rangle = 0$ for $i \neq j$, $\langle e_i, e_i \rangle = 1$ if $i = 1, \dots, p$, and $\langle e_j, e_j \rangle = -1$ if $j = p+1, \dots, n$.
(3) Fix a basis as in the previous part of the problem. Let $T \in \mathrm{End}(V)$ be an endomorphism. Prove that

$$\mathrm{tr}(T) = \sum_{i=1}^{n} \langle T(e_i), e_i \rangle.$$

Problem 2. Let G be a compact Lie group.

(1) Show that G admits a bi-invariant metric g. That is, a metric which is invariant under both right and left translations. (Hint: Argue that you can always product a left-invariant metric g_L and a left-invariant volume form $\omega \in \Omega^{\dim G}(G)$. Then, show that the metric g defined by

$$g(v, w) = \frac{1}{\int_G \omega} \int\limits_{x \in G} g_L(\mathrm{d}R_x(v), \mathrm{d}R_x(w))\omega$$

is bi-invariant.)
(2) For $h \in G$, consider the inner automorphism $\mathrm{Ad}_h \colon G \to G$ defined by $\mathrm{Ad}_h(x) = hxh^{-1}$. Show that Ad_h is an isometry with respect to the bi-invariant metric in part (a).
(3) Let $\mathfrak{g}$ be the Lie algebra of G and let ad_h be the differential of Ad_h at $e \in G$. Show that $\mathrm{ad}_h \colon \mathfrak{g} \to \mathfrak{g}$ is a linear isometry (with respect to the metric g_e on $\mathfrak{g}$.)
(4) Use part (3) to show that

$$g_e([Z, X], Y) = -g_e(X, [Z, Y])$$

for all $X, Y, Z \in \mathfrak{g}$.

Problem 3. Let (M, g) be an oriented Riemannian manifold of dimension n. Define the Riemannian volume form dvol as follows:

$$\mathrm{dvol}(v_1, \ldots, v_n) = \det(g(v_i, e_j))$$

where $\{e_i\}$ is a positively oriented orthonormal basis for T_pM.

(1) Show that the volume form is parallel.
(2) Show that in positively oriented coordinates, one has
$$\mathrm{dvol} = \sqrt{\det(g_{ij})}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$
(3) If X is a vector field, show that $L_X \,\mathrm{dvol} = \mathrm{div}(X)\,\mathrm{dvol}$.
(4) Show that the Laplacian admits the following local formula
$$\triangle u = \frac{1}{\sqrt{\det(g_{ij})}} \partial_k \left(\sqrt{\det(g_{ij})} g^{kl} \partial_l u \right)$$

Problem 4. If $F \colon M \to M$ is a diffeomorphism, and X is a vector field, define the pushforward vector field $F_* X$ by the formula

$$(F_* X)_p = \mathrm{d}F(X_{F^{-1}(p)}).$$

Show that $F_*(\nabla_X Y) = \nabla_{F_* X} F_* Y$ for any affine connection ∇.

Problem 5. Let G be a Lie group. Show that there exists a unique affine connection ∇ such that $\nabla X = 0$ for all left-invariant vector fields X. Show that this connection is torsion-free if and only if G is abelian.