MA 725 - DIFFERENTIAL GEOMETRY, I
HOMEWORK 2

Problem 1. Suppose that (M, g) is a Riemannian manifold of dimension n with constant curvature k. Recall that this means means that there exists a constant k such that for each $p \in M$ one has

$$R(v, w)z = k(v \wedge w)(z)$$

for all $v, w, z \in T_pM$.

(1) Show that (M, g) is an Einstein manifold with Einstein constant $(n-1)k$. Recall, this means that for all $p \in M$ one has

$$Ric(v, w) = (n-1)kg(v, w)$$

for all $v, w \in T_pM$.

(2) Show that (M, g) has constant scalar curvature $\mathrm{scal} = n(n-1)k$.

(3) From the scalar curvature, we can define the one-form $\mathrm{d}\,\mathrm{scal}$. Additionally, recall that given any tensor T of type $(1, r)$, we can define its divergence $\mathrm{div}(T)$ to be the $(0, r)$ tensor defined by

$$\mathrm{div}\, T = \mathrm{tr}\, \nabla T.$$

In other words, for vector fields $X_1, \ldots, X_r$, one defines $\mathrm{div}\, T$ by the rule

$$(\mathrm{div}\, T)(X_1, \ldots, X_n) = \mathrm{tr}(Y \mapsto (\nabla_Y S)(X_1, \ldots, X_n)).$$

Show that

$$\mathrm{d}\,\mathrm{scal} = 2\, \mathrm{div}\, \mathrm{Ric}\,.$$

Problem 2. Let G be a Lie group equipped with a bi-invariant metric, and identify $\mathfrak{g}$ with the Lie algebra of left-invariant vector fields. Let $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$ be the subspace of elements of the form $[X, Y]$ where $X, Y \in \mathfrak{g}$. Consider the linear map $\wedge^2\mathfrak{g} \to [\mathfrak{g}, \mathfrak{g}]$ defined by $X \wedge Y \mapsto [X, Y]$. Show that this map is an isomorphism if and only if G has constant curvature. (This only happens when the Lie algebra $\mathfrak{g}$ is $\mathfrak{su}(2)$, so the G is three-dimensional.)

Problem 3. Prove that the Ricci tensor is a symmetric $(0,2)$ tensor.

Problem 4. The *Einstein tensor* of (M, g) is defined to be the symmetric $(0,2)$ tensor

$$G \overset{\text{def}}{=} \operatorname{Ric} - \frac{\operatorname{scal}}{2} \cdot g. \tag{1}$$

Show that $G = 0$ when $\dim M = 2$. Show that, in general $\dim M > 2$, that $G = 0$ if and only if the metric is Ricci flat (meaning $\operatorname{Ric} = 0$).

Problem 5. Recall that by restriction the metric on $\mathbf{R}^n$ determines a metric on the sphere $S^{n-1}(1)$ of radius 1, which we will denote by $\mathrm{d}s^2_{n-1}$. In polar coordinates, the standard metric on $\mathbf{R}^n$ can be written as

$$g = \mathrm{d}r^2 + g_{S^{n-1}(r)} = \mathrm{d}r^2 + r^2 \mathrm{d}s^2_{n-1} \tag{2}$$

where $g_{S^{n-1}(r)} = r^2 \mathrm{d}s^2_{n-1}$ is the metric on the sphere $S^{n-1}(r)$ of radius r.

(a) Show that $\operatorname{Hess}(r) = \frac{1}{r} g_{S^{n-1}(r)}$.

More generally, let $\varphi_k(r)$ be the unique solution to the ordinary differential equation

$$\ddot{\varphi}_k + k\varphi_k = 0 \tag{3}$$

satisfying $\varphi(0) = 0, \dot{\varphi}(0) = 1$. Denote

$$g_k = \mathrm{d}r^2 + \varphi_k(r)^2 \mathrm{d}s^2_{n-1}. \tag{4}$$

(b) Show that when $k = 1$, the metric g_1 is that of the n-dimensional sphere S^n of radius 1.
(c) Show that the metric g_k is of constant curvature k.

Even more generally, consider a metric of the form

$$g_\varphi = \mathrm{d}r^2 + \varphi^2 \mathrm{d}s^2_{n-1}. \tag{5}$$

where $\varphi = \varphi(r)$ is a smooth function of r.

(d) Show that $\operatorname{Hess}(r) = (\varphi \partial_r \varphi) \mathrm{d}s^2_{n-1}$.
(e) Show that the only metrics g_φ which are Ricci flat are, in fact, flat metrics (so $\varphi(r) = a \pm r$).