MA 725 - DIFFERENTIAL GEOMETRY, I

HOMEWORK 3

Problem 1. Clean up.

(1) Fix a connection on a vector bundle and let ω, Ω be the connection and curvature matrices relative to a local frame. Prove the generalized Bianchi identity

$$\mathrm{d}\Omega^k = \Omega^k \wedge \omega - \omega \wedge \Omega^k \tag{1}$$

(2) Suppose that ∇, ∇' are connections. Show that for any numbers t, s such that $t + s = 1$ that the linear operator

$$t\nabla + s\nabla' \tag{2}$$

is a connection.

Problem 2.

(1) Show that $P \in \mathbf{R}[\mathfrak{gl}_r]$ is an invariant polynomial if and only if $P(XY) = P(YX)$ for all matrices X, Y.

(2) Let $\{s_i\}$, $\{\sigma_i\}$ be the trace and symmetric polynomials, respectively. Show that

$$s_1 = \sigma_1, \quad s_2 = \sigma_1^2 - 2\sigma_2, \quad s_3 = \sigma_1^3 - 3\sigma_1\sigma 2 + 3\sigma_3$$

(3) Suppose that $V \to M$ is a complex vector bundle. Show that

$$p_1(V) = c_1(V)^2 - 2c_2(V). \tag{3}$$

Problem 3. In this problem you will compute the Chern classes of complex projective space.

(1) Let γ be the canonical line bundle over $\mathbf{CP}^n$. Describe a vector bundle ω with the property that $\gamma \oplus \omega$ is isomorphic to the trivial rank $(n+1)$ complex vector bundle.

(2) Show that $\mathrm{Hom}_{\mathbf{C}}(\gamma, \omega) \cong T_{\mathbf{CP}^n}$.

(3) Let $\underline{\mathbf{C}}$ be the trivial rank one vector bundle. Show that $\mathrm{T}_{\mathbf{CP}^n} \oplus \underline{\mathbf{C}} \cong (\gamma^*)^{\oplus(n+1)}$.

(4) Conclude that

$$c(\gamma^*)^{n+1} = (1 - c_1(\gamma))^{n+1}. \tag{4}$$

(5) Conclude that $c(T_{\mathbf{CP}^n}) = (1+a)^{n+1}$ for some $a \in H^2(\mathbf{CP}^n; \mathbf{Z})$.

Problem 4. Suppose that E is a $2k$-dimensional vector bundle. Show that $p_k(E) = e(E)^2$.

Problem 5. Suppose that L is a complex line bundle and V is a complex vector bundle of rank r. Find a formula for $c(L \otimes V)$ in terms of $c(L)$ and $c(V)$.

Problem 6. Suppose that S^n admits a complex structure. Show that $n \equiv 2 \mod 4$ (there are no "big theorems" at play here).