Theorem : [The fundamental theorem of Riem geometry]

The linear map

$$\nabla : \Gamma(TM) \longrightarrow \Gamma(TM \otimes T^*M)$$

is the unique one s.t.

① $\nabla_Y (fX) = (Y \cdot f) X + f \nabla_Y X$

(Derivation)

②

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

(torsion-free).

③ $Z \cdot g(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$.

(preserves metric).

Pf: ① Note $L_{fX}\, g = f L_X\, g$

and $(fX)^\flat = f X^\flat$.

② We establish Koszul's formula:

$$2g(\nabla_Y X, Z) = (L_X g)(Y,Z) + (dX^\flat)(Y,Z)$$

$$= X\cdot g(Y,Z) - g([X,Y],Z) - g(Y,[X,Z])$$

$$+ Y\cdot X^\flat(Z) - Z\cdot X^\flat(Y) - X^\flat([Y,Z])$$

$$= X\cdot g(Y,Z) - g([X,Y],Z) - g(Y,[X,Z])$$

$$+ Y\cdot g(X,Z) - Z\cdot g(X,Y) - g(X,[Y,Z])$$

$$= X\cdot g(Y,Z) + Y\cdot g(X,Z) - Z\cdot g(X,Y)$$

$$- g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y)$$

Now, using this:

$$2g(\nabla_Y X - \nabla_X Y, Z) =$$

$$X\cdot g(Y,Z) + Y\cdot g(X,Z) - Z\cdot g(X,Y)$$

$$- g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y)$$

$$- Y\cdot g(X,Z) - X\cdot g(Y,Z) + Z\cdot g(X,Y)$$

$$- g([X,Y],Z) + g([X,Z],Y) - g([Z,Y],X)$$

$= -2g([X,Y], Z)$.

This proves ② . (torsion-free).

<u>Next</u> : $2g(\nabla_Z X, Y) + 2g(X, \nabla_Z Y)$

$= 2Z \cdot g(X,Y)$.

by using Koszul again ... So this is ③.

Conversely, if we have ∇' satisfying same axioms

then

$2g(\nabla_Y X, Z) =$

$X \cdot g(Y,Z) + Y \cdot g(X,Z) - Z \cdot g(X,Y)$

$- g([X,Y], Z) - g([Y,Z], X) + g([Z,X], Y)$

$= 2g(\nabla'_Y X, Z)$

$\Rightarrow \quad \nabla = \nabla'$. $\blacksquare$

Dfn: An affine connection is a linear operator

$$\nabla : \Gamma(TM) \longrightarrow \Gamma(TM \otimes T^*M)$$

s.t. $\nabla_Y (fX) = (Y \cdot f) X + f \nabla_Y X$

(Derivation).

An equivalent way to state theorem:

Thm: On any Riem. manifold there is a unique affine connection ∇ which satisfies

1) Torsion-free: $\nabla_X Y - \nabla_Y X = [X, Y]$

2) Preserves connection:

$$g(\nabla_Z X, Y) + g(X, \nabla_Z Y) = Z \cdot g(X, Y).$$

∇ is called the Levi-Civita connection.

Lemma: Sps $v \in T_pM$ and let
X, Y be v.f's s.t.
$X\big|_U = Y\big|_U$ for some nbd U of p.

Then $\nabla_v X = \nabla_v Y$.

Pf: Let $\lambda \in C^\infty(M)$ s.t. $\lambda \equiv \begin{cases} 0 & \text{on } M - U \\ 1 & \text{on some small } p \in V \subset U \end{cases}$

So, $\lambda X = \lambda Y$ on M. Thus

$$\nabla_v (\lambda X)\big|_p = \lambda(p)\, \nabla_v X\big|_p + (d\lambda)(v) \cdot X(p)$$

$$= \nabla_v X$$

since $d\lambda\big|_p = 0$ and $\lambda(p) = 1$. So

$$\nabla_v X = \nabla_v(\lambda X) = \nabla_v(\lambda Y) = \nabla_v Y. \qquad \blacksquare$$

· Derivatives of tensors

If S is a $(1,1)$ tensor field then we can still define $\nabla_X S$. We require

$$\nabla_X (S(Y)) = (\nabla_X S)(Y) + S(\nabla_X Y).$$

In other words, ∇S is the $(1,2)$ tensor

$$\nabla S(X, Y) = (\nabla_X S)(Y)$$
$$= \nabla_X (S(Y)) - S(\nabla_X Y)$$

More generally, if S is type $(1, r)$, then let ∇S be the type $(1, r+1)$ tensor defined by

$$(\nabla S)(X, Y_1, \dots Y_r) = (\nabla_X S)(Y_1, \dots Y_r)$$
$$= \nabla_X (S(Y_1, \dots Y_r)) - \sum_{i=1}^{r} S(Y_1, \dots, \nabla_X Y_i \dots Y_r)$$

- We interpret $\nabla_X f = X \cdot f$, when $f \in C^\infty(M)$.

We can extend this to tensors of type $(0,r)$ by the formula

$\nabla \alpha$ is type $(0, r+1)$

$$(\nabla \alpha)(X, Y_1, \dots, Y_r) = X \cdot \alpha(Y_1, \dots Y_r) - \sum_i \alpha(Y_1, \dots, \nabla_X Y_i, \dots Y_r).$$

- For example, when g is a metric then ∇g is the $(0,3)$ tensor

$$(\nabla g)(X, Y_1, Y_2) = X \cdot g(Y_1, Y_2) - g(\nabla_X Y_1, Y_2) - g(Y_1, \nabla_X Y_2).$$

So ∇ preserves metric $\Leftrightarrow$ $\nabla g = 0$.

Dfn: A tensor S is parallel if $\nabla S \equiv 0$.

• More notations.

Dfn: For $f \in C^\infty(M)$ define

$$\mathrm{Hess}(f) \in \Gamma(T^{*\otimes 2}) = \Gamma(T(0,2))$$

to be the symmetric $(0,2)$ tensor $\frac{1}{2} L_{\nabla f}\, g$.

Lem: On $M = \mathbb{R}^n$ have

$$\mathrm{Hess}(f) = \partial_i \partial_j f \, dx^i dx^j$$

• There is a related $(1,1)$ tensor S_f where

$$S_f(X) = \nabla_X \nabla f.$$

Have $\quad \mathrm{Hess}\, f(X,Y) = g(S_f(X), Y).$

$$\Big[\; 2g(S_f(X), Y) = 2g(\nabla_X \nabla f, Y)$$

$$= (L_{\nabla f}\, g)(Y,Z) + d(\nabla f)^\flat(Y,Z)$$

$$= 2\,\mathrm{Hess}\, f(X,Y). \quad \Big]$$

We have used: $d(\nabla f)^\flat = 0$.

$\left[(\nabla f)^\flat = df \quad \text{and} \quad d^2 = 0. \right]$.

· Observe $T(1,1) = T_M \otimes T_M^* = \mathrm{End}(T_M)$.

There is a bundle map $\mathrm{tr} : \mathrm{End}(T_M) \to \underline{\mathbb{R}}$

trivial bundle.

So, if $S \in \mathcal{T}(1,1) \Rightarrow \mathrm{tr}(S) \in C^\infty(M)$.

<u>Dfn</u>: 1) The Laplacian of f is:

$$\Delta f = \mathrm{tr}\, S_f \in C^\infty(M)$$

2) The divergence of X is:

$$\mathrm{div}\, X = \mathrm{tr}\, \nabla X \in C^\infty(M).$$

<u>Lem</u>: $\Delta f = \mathrm{div}(\nabla f)$.

- Local coordinates. Recall, in local coords

$$g(X,Y) = g_{ij} X^i Y^j .$$

Note $X^\flat = g(X, \cdot) = g_{ij}\, dx^i(X)\, dx^j(\cdot)$

$$= g_{ij} X^i dx^j .$$

We denote the inverse matrix to $[g_{ij}]$ by $[g^{ij}]$.

$$g^{ik} g_{kj} = \delta^i_j .$$

Sps $\theta = \theta_j dx^j$, is dual to $X = X^i \partial_i$.

That is $X^\flat = \theta$.

Then: $$X^k = g^{kj}\theta_j$$

and $$\theta_j = g_{kj} X^k .$$

- $\nabla f = g^{ij} \partial_i f \, \partial_j$, $df = \partial_j f \, dx^j$.

- Local formula for L.C. connection.

$$\nabla_Y X = \nabla_{Y^i \partial_i} (X^j \partial_j)$$

$$= Y^i \nabla_{\partial_i} (X^j \partial_j)$$

$$= Y^i \left(\partial_i X^j \partial_j + X^j \nabla_{\partial_i} \partial_j \right).$$

Denote $\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k$.

Thus: $g(\nabla_{\partial_i}\partial_j, \partial_l) = g(\Gamma^k_{ij}\partial_k, \partial_l)$

$$= \Gamma^k_{ij}\, g_{kl}.$$

$$\frac{1}{2}\left\{ (L_{\partial_j} g)(\partial_i, \partial_l) + d(\partial_j^\flat)(\partial_i, \partial_l) \right\}$$

$$= \frac{1}{2}\left\{ \partial_j g_{il} + \partial_i \left(\partial_j^\flat (\partial_l) \right) - \partial_l \left(\partial_j^\flat (\partial_i) \right) \right\}$$

$$= \frac{1}{2}\left\{ \partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ji} \right\}$$