• Local coordinates. Recall, in local coords

$$g(X,Y) = g_{ij} X^i Y^j .$$

Note $$X^\flat = g(X, \cdot) = g_{ij}\, dx^i(X)\, dx^j(\cdot)$$
$$= g_{ij} X^i dx^j .$$

We denote the inverse matrix to $[g_{ij}]$ by $[g^{ij}]$.

$$g^{ik} g_{kj} = \delta^i_j .$$

Sps $\theta = \theta_j dx^j$, is dual to $X = X^i \partial_i$.

That is $X^\flat = \theta$.

Then: $$X^k = g^{kj} \theta_j$$
and
$$\theta_j = g_{kj} X^k .$$

• $\nabla f = g^{ij} \partial_i f \partial_j$, $df = \partial_j f\, dx^j$.

- Local formula for L.C. connection.

$$\nabla_Y X = \nabla_{Y^i \partial_i}(X^j \partial_j)$$

$$= Y^i \nabla_{\partial_i}(X^j \partial_j)$$

$$= Y^i \left(\partial_i X^j \partial_j + X^j \nabla_{\partial_i} \partial_j \right).$$

Denote $\quad \nabla_{\partial_i} \partial_j = \Gamma^k_{ij}\, \partial_k .$

Thus: $\quad g(\nabla_{\partial_i}\partial_j, \partial_l) = g(\Gamma^k_{ij}\partial_k, \partial_l)$

$$= \Gamma^k_{ij}\, g_{kl} .$$

$$\frac{1}{2}\Big\{ (L_{\partial_j} g)(\partial_i, \partial_l) + d(\partial_j^\flat)(\partial_i, \partial_l)$$

$$= \frac{1}{2}\Big\{ \partial_j g_{il} + \partial_i\big(\partial_j^\flat(\partial_l)\big) - \partial_l\big(\partial_j^\flat(\partial_i)\big)\Big\}$$

$$= \frac{1}{2}\Big\{ \partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ji} \Big\}$$

Thus:

$$\boxed{\Gamma^{m}_{ij} = \frac{1}{2}\left(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{il}\right) g^{lm}.}$$

- Γ^{k}_{ij} is called the "Christoffel symbol" (of the second kind). Related, is the expression

$$\Gamma_{ij,k} = \frac{1}{2}\left(\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij}\right)$$

$$= g\left(\nabla_{\partial_i} \partial_j , \partial_k\right).$$

Note $\Gamma^{k}_{ij} = g^{kl}\, \Gamma_{ij,l}$.

- It is always possible to choose coordinates $\{x^i\}$ near $p \in M$ s.t.

$$\begin{cases} g_{ij}\big|_p = \delta_{ij} \\ \partial_k g_{ij}\big|_p = 0 . \end{cases}$$

Called "normal coordinates".

Thus, in these coordinates $\Gamma^k_{ij}\big|_p = 0$ and thus

$$\nabla_Y X = Y^i(p)\, \partial_i X^j \big|_p\, \partial_j .$$

in normal coordinates.

(See HW ...) //

- Torsion-free $\Leftrightarrow$ $\boxed{\Gamma^k_{ij} = \Gamma^k_{ji}.}$

$$\Gamma^k_{ij}\,\partial_k = \nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i = \Gamma^k_{ji}\,\partial_k .$$

- Metric compatibility:

$$\partial_k g_{ij} = g(\nabla_{\partial_k}\partial_i, \partial_j) + g(\partial_i, \nabla_{\partial_k}\partial_j).$$

$$= \Gamma_{ki,j} + \Gamma_{kj,i} .$$

$\Rightarrow$ Christoffel symbols determine all derivatives of the metric.

- Can obtain more local formulas.

<u>Prop</u>:

$$\mathrm{Hess}\, f\,(\partial_i, \partial_j)$$

$$= \partial_i \partial_j f - \Gamma^k_{ij} \partial_k f$$

- Curvature : Let

$$R \in \Gamma(T \otimes (T^*)^{\otimes 3})$$
$$\shortparallel$$
$$\mathcal{T}(1,3)$$

defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Why is this tensorial in "Z"? For $f \in C^\infty$:

$$R(X,Y)(fZ) = \nabla_X \nabla_Y (fZ) - \nabla_Y \nabla_X (fZ) - \nabla_{[X,Y]}(fZ)$$

$$= \nabla_X \big((Y\cdot f) Z + f \nabla_Y Z \big)$$

$$- \nabla_Y \big((X \cdot f) Z + f \nabla_X Z \big)$$

$$- \big([X,Y]\cdot f + f \nabla_{[X,Y]} Z \big).$$

$$= f\, R(X,Y)Z$$

$$+\; X\cdot(Y\cdot f) - Y\cdot(X\cdot f) - [X,Y]\cdot f$$

$$= f\, R(X,Y)Z\,. \quad \checkmark$$

$$\begin{tikzcd}
R \arrow[d] & \in \Gamma\big(\overbrace{T^*\otimes T^*}^{\text{skew symm.}}\otimes T^*\otimes T\big) \arrow[d, "g\ \cong"'] \\
\tilde{R} & \in \Gamma\big(\underbrace{T^*\otimes T^*}_{\text{skew symm}}\otimes \underbrace{T^*\otimes T^*}_{\text{skew symm}}\big).
\end{tikzcd}$$

$$\tilde{R}(X,Y,Z,W)\,.$$