Prop: 1) $R(X,Y,Z,W) = -R(Y,X,Z,W)$

$= +R(Y,X,W,Z)$.

2) $R(X,Y,Z,W) = R(Z,W,X,Y)$.

3) (Bianchi 1)

$$R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0.$$

4) (Bianchi 2)

$$(\nabla_Z R)(X,Y) + (\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) = 0.$$

Pf: 1) Have already seen $R(X,Y,Z,W) = -R(Y,X,Z,W)$.

Now: $g(R(X,Y)Z, Z) = g(\nabla_X \nabla_Y Z, Z) - g(\nabla_Y \nabla_X Z, Z) - g(\nabla_{[X,Y]} Z, Z)$

$= X \cdot g(\nabla_Y Z, Z) - g(\nabla_Y Z, \nabla_X Z)$

$- Y \cdot g(\nabla_X Z, Z) + g(\nabla_X Z, \nabla_Y Z)$

$- \frac{1}{2}[X,Y] \cdot g(Z,Z)$

$= \frac{1}{2} X \cdot (Y \cdot g(Z,Z)) - \frac{1}{2} Y \cdot (X \cdot g(Z,Z)) - \frac{1}{2} [X,Y] \cdot g(Z,Z)$

$= 0$ (def^n of $[X,Y]$).

So: $g(X,Y,Z,W) + g(X,Y,W,Z)$

$= g(X,Y,Z+W,Z+W) = 0$.

3) Let T be a mapping w/ 3 inputs, set

$\sigma T(X,Y,Z) =$ cyclic perm.

(e.g. $\sigma [X,[Y,Z]] = 0$ is Jacobi)

Now

$\sigma R(X,Y)Z = \sigma \nabla_X \nabla_Y Z - \sigma \nabla_Y \nabla_X Z - \sigma \nabla_{[X,Y]} Z$

$= \sigma \nabla_Z \nabla_X Y - \sigma \nabla_Z \nabla_Y X - \sigma \nabla_{[X,Y]} Z$

$= \sigma \nabla_Z (\nabla_X Y - \nabla_Y X) - \sigma \nabla_{[X,Y]} Z$

$= \sigma \nabla_Z [X,Y] - \sigma \nabla_{[X,Y]} Z$

$= \sigma [X,[Y,Z]] = 0$.

2) Use 1) and 3)

$$R(X,Y,Z,W) = -R(Z,X,Y,W) - R(Y,Z,X,W)$$
$$= R(Z,X,W,Y) + R(Y,Z,W,X)$$
$$= -R(W,Z,X,Y) - R(X,W,Z,Y)$$
$$- R(W,Y,Z,X) - R(Z,W,Y,X)$$
$$= 2R(Z,W,X,Y) + R(X,W,Y,Z)$$
$$+ R(W,Y,X,Z)$$
$$= 2R(Z,W,X,Y) - R(X,Y,Z,W)$$

4) Note $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$

So: $(\nabla_Z R)(X,Y)W = \nabla_Z(R(X,Y)W) - R(\nabla_Z X, Y)W$
$$- R(X,\nabla_Z Y)W - R(X,Y)\nabla_Z W$$
$$= [\nabla_Z, R(X,Y)]W$$
$$- R(\nabla_Z X, Y)W - R(X,\nabla_Z Y)W.$$

Now cyclically permute and cancel terms!

(Exercise) ☒

• Curvature operator.

Let $\Lambda^2 T \subset T(2,0)$ "bivectors".

Define a metric on $\Lambda^2 T$ by the rule:

$$g(x \wedge y, v \wedge w) = \det \begin{pmatrix} g(x,v) & g(x,w) \\ g(y,v) & g(y,w) \end{pmatrix}.$$

//

Using an inner product on a v.s. V, we can see

$$\Lambda^2 V \hookrightarrow \operatorname{End} V$$

$$v \wedge w \longmapsto \left(z \longmapsto \langle w, x \rangle v - \langle v, x \rangle w \right)$$

skew symmetric endomorphisms.

Note: $(x \wedge y)(z) + (y \wedge z)(x) + (z \wedge x)(y) = 0$

is a Jacobi indentity. //

- Note, we can view:

$$R \in \Lambda^2 T^* \otimes \Lambda^2 T^*.$$

Use
$$\begin{array}{ccc} R & \in & \Lambda^2 T^* \otimes \Lambda^2 T^* \\ \big\downarrow & & \qquad \wr\| \;\; g \\ \mathcal{R} & \in & \Lambda^2 T^* \otimes \Lambda^2 T \end{array}$$

"curvature operator".

$$g(\mathcal{R}(X\wedge Y), V\wedge W) = R(X\wedge Y, V\wedge W).$$

- We will say that (M,g) has <u>constant curvature</u> if $\exists$ k = constant s.t. $\forall$ $p\in M$ and $\pi \in \Lambda^2 T_pM$ one has

$$\mathcal{R}(\pi) = k\cdot\pi.$$

curvature operator $\Lambda^2T_pM \longrightarrow \Lambda^2T_pM$

<u>Prop</u>: (M,g) has constant curvature k $\Leftrightarrow$

$\forall$ $p\in M$, $v_1, v_2, v_3 \in T_pM$ one has

$$R(v_1,v_2)v_3 = k(v_1\wedge v_2)(v_3)$$
$$= k\big(g(v_2,v_3)v_1 - g(v_1,v_3)v_2\big).$$