MA 725

Differential Geometry, I.

- We will assume familiarity w/ the theory of smooth manifolds. [See, for example, the textbook of Lee].

Differential geometry is the study of smooth manifolds equipped w/ additional structure: a Riemannian metric.

$\rightsquigarrow$ Concepts of length, angle, volume ...

Dfn: An inner product on a real vector space V is a symmetric bilinear map

$$\langle -,- \rangle : V \times V \longrightarrow \mathbb{R}$$

such that:

- $\langle v, v \rangle \geq 0$ for all $v \in V$ and if $\langle v, v \rangle = 0$ then $v = 0$.

(non-degenerate).

Ex: Let $\{e_i\}$ be a basis for V.

Define $\langle -,- \rangle$ on basis elements by

$$\langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

This is the "standard inner product" or "dot product" associated to a basis.

Lem: If $\langle -,- \rangle$ is an inner product on V and $W \subset V$ is a subspace, then

$$\langle -,- \rangle \Big|_{W \times W} : W \times W \to \mathbb{R}$$

is an inner product on W.

- Inner products and matrices.

Again, let $\{e_i\}$ be a basis for V.

The inner product $\langle -,- \rangle$ is determined by its values on pairs of basis elements.

Sps

$$\langle e_i , e_j \rangle =: b_{ij} \in \mathbb{R}.$$

Then we can think of $B = (b_{ij})$ as a $n \times n$ matrix. By assumption $b_{ij} = b_{ji}$, so B is symmetric, $B^T = B$.

<u>Prop</u>: The matrix $B = (b_{ij} = \langle e_i, e_j \rangle)$ is positive definite, in the sense

$$\underline{x}^T B \underline{x} > 0 \quad \text{for all } \underline{x} \in \mathbb{R}^n \setminus \{0\}.$$

In fact, there is a bijection

$$\left\{ \begin{matrix} \text{Inner products} \\ \text{on v.s. } V \\ \text{of dim} = n \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{non-deg, symmetric} \\ n \times n \text{ matrices} \\ \text{w/ real coeffs.} \end{matrix} \right\}.$$

<u>Pf</u>: Easy. $\blacksquare$

- The $\otimes$ - Hom adjunction formula gives

$$\begin{array}{c} \mathrm{Hom}_{\text{bilinear}}(V \times V, \mathbb{R}) \ni \langle -,- \rangle \\ \text{\rotatebox{90}{$\cong$}} \\ \mathrm{Hom}(V \otimes V, \mathbb{R}) \cong \mathrm{Hom}(V, V^\vee) \end{array}$$

The condition that $\langle -,- \rangle$ is nondegenerate is equivalent to the condition that the corresponding linear map $V \to V^\vee$ is an *isomorphism*.

Going further, we see that $\langle -,- \rangle$ defines a vector in $V^\vee \otimes V^\vee$. A basis for this vector space is $\{e^i \otimes e^j\}_{i,j}$ where $\{e^i\}$ is the dual basis to $\{e_i\}$. Therefore

$$\langle -,- \rangle = \sum_{i,j} b_{ij} \, e^i \otimes e^j$$

where $b_{ij} = \langle e_i, e_j \rangle$.

Sps $(V, \langle -,- \rangle)$ is an inner product space.

Some additional concepts:

① A linear isometry $F : (V, \langle -,- \rangle_V) \to (W, \langle -,- \rangle_W)$ is a linear map s.t.

$$\langle F(v), F(w) \rangle_W = \langle v, w \rangle_V$$

for all $v, w \in V$.

② The set of bijective linear isometries $V \to V$ forms a group that is denoted

$$O(V) = \text{"orthogonal group"}.$$

③ If $W \subset V$ is a subspace. Let

$$W^\perp = \{ v \in V \mid \langle v, w \rangle = 0 \ \forall w \in W \}.$$

"The perpendicular space".

• Manifolds. Let M be a smooth manifold and TM is its tangent bundle. A Riemannian metric will be the data of an inner product on T_pM for each $p \in M$.

The technical thing is that we require that $\langle -,- \rangle_p$ vary smoothly as a function of p.

Dfn: A Riemannian metric on M is an inner product $\langle -,- \rangle_p$ on T_pM for each p, such that for all C^∞-vector fields X, Y the function

$$M \ni p \longmapsto \langle X_p, Y_p \rangle_p \in \mathbb{R}$$

is smooth (C^∞).

Rmk: We will go back and forth between the notations

$$\langle -,- \rangle \longleftrightarrow g$$

for a Riem. metric.

Exercise: A Riem. metric determines a smooth section of $T^*M \otimes T^*M$.

- In local coordinates we can express:

$$g = g_{ij} \, dx^i \cdot dx^j$$

where:

- $\{x^i\}$ is a local coordinate.
- $\{\partial_i\}, \{dx^i\}$ are the corresponding local frames for TM, T^*M respectively.
- $g_{ij} = g(\partial_i, \partial_j)$.

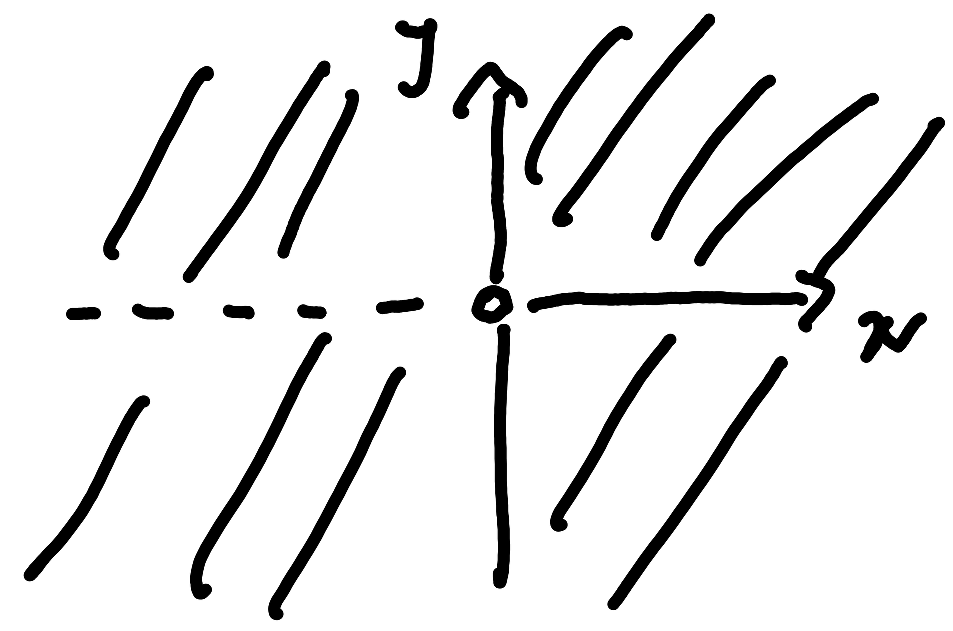
Ex: The "canonical" metric on $\mathbb{R}^n$ is

$$g_{can} = \delta_{ij} \, dx^i \cdot dx^j = \sum_i dx^i \cdot dx^i$$

We are using Einstein sumation convention of repeated indices. So, this is

$$\sum_{i,j} \delta_{ij} \, dx^i \cdot dx^j = \sum_i dx^i \cdot dx^i$$

Ex: $M = \mathbb{R}^2 - \{\theta = \pi\}$

$=$ 

we have polar coordinates $x = r\cos\theta$, $y = r\sin\theta$.

In polar coordinates, the canonical/flat metric is

$$dr^2 + r^2 d\theta^2.$$

That is: $g_{rr} = 1$, $g_{r\theta} = g_{\theta r} = 0$
$g_{\theta\theta} = r^2$.

To see this, note:
$$dx = \cos\theta\, dr - r\sin\theta\, d\theta$$
$$dy = \sin\theta\, dr + r\cos\theta\, d\theta$$

$$dx^2 = \cos^2\theta\, dr^2 + r^2\sin^2\theta\, d\theta^2 - r\cos\theta\sin\theta\, dr\, d\theta$$
$$dy^2 = \sin^2\theta\, dr^2 + r^2\cos^2\theta\, d\theta^2 + r\cos\theta\sin\theta\, dr\, d\theta$$

- Let (M, g) be a Riemannian manifol

(= a manifold equipped w/ a Riem metric.)

If N is another manifold, and

$$F : N \longrightarrow M$$

is a smooth map, we get new metric F^*g on N defined by:

$$(F^*g)(v, w) = g(dF(v), dF(w)).$$

- A <u>Riemannian map</u> is a smooth map

$$F : (M, g_M) \longrightarrow (N, g_N)$$

s.t. $g_M = F^* g_N$.

· A <u>diffeomorphism</u> $F : M \to M$ for which

$F^* g = g$ is called an <u>isometry</u>.

<u>Dfn</u>: Define the group of isometries

$$\mathrm{Isom}(M, g) = \left\{ F : M \to M \;\middle|\; \begin{array}{l} F \text{ diffeo} \\ F^* g = g \end{array} \right\}.$$

A <u>Riemannian</u> <u>immersion</u> is an immersion

$$\gamma : M \longrightarrow N$$

s.t. $\gamma^* g_N = g_M$.

<u>Ex</u>: What are the Riemannian immersions

$$\gamma : \mathbb{R} \hookrightarrow \mathbb{R}^2 \quad ?$$

where we use the flat metric for both $\mathbb{R}$ and $\mathbb{R}^2$.

- lines, of course...
- γ curve of constant speed $|\gamma'(t)| = 1$.

$\rightsquigarrow$ $\gamma(t) = (\cos t, \sin t)$, $t \in \mathbb{R}$

for example.

For a Riemannian embedding

$$\gamma : \mathbb{R} \hookrightarrow \mathbb{R}^2$$

consider

$$\gamma(t) = \left(\log\left(t + \sqrt{1+t^2}\right), \sqrt{1+t^2} \right).$$

$$\frac{d}{dt} \log\left(t + \sqrt{1+t^2}\right) = \frac{1 + \frac{t}{\sqrt{1+t^2}}}{t + \sqrt{1+t^2}} = \frac{1}{\sqrt{1+t^2}}.$$

So

$$\left(\gamma^* g_{std}\right)(t) = \left(\frac{1}{1+t^2} + \frac{t^2}{1+t^2} \right) dt^2$$

$$= dt^2.$$

So γ preserves the standard metrics.

- A <u>Riemannian</u> <u>submersion</u> is a submersion

$$F : M \longrightarrow N$$

s.t.

$$DF_p\Big| : \left(\ker DF_p\right)^\perp \longrightarrow T_{F(p)} N$$

is a linear isometry.

$\underline{Ex}$: Consider the embedding:

$$\begin{array}{ccc} i : S^n & \hookrightarrow & \mathbb{R}^{n+1}. \\ \| & & \\ \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \} & & \end{array}$$

Then, from $g_{std} = \sum_i (dx^i)^2$, we get a metric $i^* g_{std}$ on S^n.

$\underline{Ex}$: Use complex coordinates to write

$$\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{C} \times \mathbb{C} = \mathbb{C}^2$$

$$\underset{\underline{x}}{\underbrace{(x_1, y_1, x_2, y_2)}} \longleftrightarrow (z_1, z_2) = (x_1 + iy_1, x_2 + iy_2).$$

Note that $\|\underline{x}\|^2 = |z_1|^2 + |z_2|^2 = z_1\bar{z}_1 + z_2\bar{z}_2$.

So

$$\begin{array}{ccc} S^3(1) = \{ \|\underline{x}\|^2 = 1 \} & \subset & \mathbb{R}^4 \\ \| & & \| \\ \{ |z_1|^2 + |z_2|^2 = 1 \} & \subset & \mathbb{C}^2. \end{array}$$

$$S^2\left(\tfrac{1}{2}\right) = \left\{ x^2 + z\bar{z} = 1 \right\} \subset \begin{array}{c} \mathbb{R}^3 \\ \parallel \\ \mathbb{R} \times \mathbb{C} \end{array}.$$

Define

$$F : S^3(1) \longrightarrow S^2\left(\tfrac{1}{2}\right)$$

$$F(z_1, z_2) = \left(\tfrac{1}{2}\left(|z_1|^2 - |z_2|^2\right),\ z_1 \bar{z}_2 \right).$$

(Check this is well-defined)

We will shortly prove that this F is a Riem. submersion.