- Let (M, g) be a Riemannian manifol
  (= a manifold equipped w/ a Riem metric.)

If N is another manifold, and

$$F : N \longrightarrow M$$

is a smooth map, we get new metric F^*g on N defined by:

$$(F^*g)(v, w) = g\big(dF(v), dF(w)\big).$$

- A Riemannian map is a smooth map

$$F : (M, g_M) \longrightarrow (N, g_N)$$

s.t. $g_M = F^* g_N$.

· A diffeomorphism $F : M \to M$ for which

$F^* g = g$ is called an isometry.

Dfn: Define the group of isometries

$$\mathrm{Isom}(M, g) = \left\{ F : M \to M \;\middle|\; \begin{array}{l} F \text{ diffeo} \\ F^* g = g \end{array} \right\}.$$

A Riemannian immersion is an immersion

$$\gamma : M \longrightarrow N$$

s.t. $\gamma^* g_N = g_M$.

Ex: What are the Riemannian immersions

$$\gamma : \mathbb{R} \hookrightarrow \mathbb{R}^2 \quad ?$$

where we use the flat metric for both $\mathbb{R}$ and $\mathbb{R}^2$.

- lines, of course...
- γ curve of constant speed $|\gamma'(t)| = 1$.

$\rightsquigarrow$ $\gamma(t) = (\cos t, \sin t)$, $t \in \mathbb{R}$

for example.

For a Riemannian embedding

$$\gamma : \mathbb{R} \hookrightarrow \mathbb{R}^2$$

consider

$$\gamma(t) = \left(\log\left(t + \sqrt{1+t^2}\right), \sqrt{1+t^2} \right).$$

$$\frac{d}{dt} \log\left(t + \sqrt{1+t^2}\right) = \frac{1 + \frac{t}{\sqrt{1+t^2}}}{t + \sqrt{1+t^2}} = \frac{1}{\sqrt{1+t^2}}.$$

So

$$(\gamma^* g_{std})(t) = \left(\frac{1}{1+t^2} + \frac{t^2}{1+t^2} \right) dt^2$$

$$= dt^2.$$

So γ preserves the standard metrics.

- A Riemannian submersion is a submersion

$$F : M \to N$$

s.t.

$$DF_p\Big| : (\ker DF_p)^\perp \longrightarrow T_{F(p)} N$$

is a linear isometry.

- <u>Coordinates</u>: We have seen that in local coordinates a metric looks like

$$g = \underbrace{g(\partial_i, \partial_j)}_{g_{ij}} dx^i dx^j .$$

Similarly, we get local expressions for g by using <u>frames</u>.

A local frame on $U \subset M$ (for TM) is a collection of vector fields $\{X_i\}$ on U s.t. $\{X_i|_p\}$ are linearly independent for every $p \in U$.

Given a local frame, let $\{\sigma^i\}$ be its dual local frame for T^*M. So, σ^i is a 1-form. Then

$$g = \underbrace{g(X_i, X_j)}_{g_{ij}} \sigma^i \sigma^j$$

- Connections

Given a fn $f : M \to \mathbb{R}$, we know that its differential

$$df : TM \longrightarrow \mathbb{R}$$

or

$$df \in \Omega^1(M)$$ is a measure of the "change" of f.

In local coordinates $df = \dfrac{\partial f}{\partial x^i} \, dx^i$.

- If M is equipped w/ a Riem metric g, we can define another "derivative" of a fn.

Let $\operatorname{grad} f = \nabla f$ be the vector field s.t.

$$g(v, \nabla f) = df(v) \left(\stackrel{\text{def}}{=} \nabla_v f \right).$$

for all $v \in \Gamma(TM)$.

In standard metric g_{std} on $\mathbb{R}^n$, one has

$$\nabla f = \delta^{ij} \frac{\partial f}{\partial x^i} \partial_j = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \partial_i .$$

This clearly depends on the metric and the coordinate we choose!

$\underset{\sim}{\text{So}}$: df is an invariant quantity (does not depend on metric)

∇f is $\underline{\underline{\text{not}}}$.

- $\underline{Q}$: How do we differentiate vector fields?

First, we need to point out an important feature of Riem. metrics. They determine an isomorphism

$$\Gamma(M, TM) \overset{\cong}{\longleftrightarrow} \Omega^1(M) = \Gamma(M, T^* M)$$

$$X \longmapsto i_X g = X^\flat$$

"Musical isomorphism".

Dfn: On $\mathbb{R}^n$, the covariant derivative of a vector field $X = a^i \partial_i$ in the direction of $Y = b^i \partial_i$ is the vector field

$$\nabla_Y X = (\nabla_Y a^i)\, \partial_i$$

$$= (d a^i)(Y)\, \partial_i .$$

This expression depends on the chosen coordinate!

- Recall, the Lie derivative along a v.f. is defined in a variety of situations.

f function $\rightsquigarrow$ $L_X f = X \cdot f$

α differential form $\rightsquigarrow$ $L_X \alpha = d i_X \alpha + i_X d\alpha$.

More generally, L_X makes sense when acting on tensor fields.

Let $T(r,s)$ denote the vector bundle

$$(TM)^{\otimes r} \otimes (T^*M)^{\otimes s}.$$

So, local sections are linear combos of

$$f(x)\, \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$$

- Given a v.f. X, let Φ_t^X be its local flow. So, for each $p \in M$ we have

$$X_p = \left(\Phi^X(p)\right)'(0).$$

In particular, for each $t \in$ flow domain, $\Phi_t^X : M \to M$ is a local diffeomorphism.

Given a local diffeo, we can pullback both covariant / contravariant tensors.

e.g. for covariant we use

$$(D_p \Phi_t^X)^{-1} : T_{\Phi_t^X(p)} M \to T_p M.$$

If $\mathcal{T}$ is section of $T(r,s)$, define

$$(L_X \mathcal{T})(p) = \frac{d}{dt}\Big|_{t=0} (\Phi^X_t)^* \mathcal{T} \,\Big|_p .$$

- More practically $L_X \mathcal{T}$ can be defined algebraically.

① $L_X(\mathcal{T} \otimes S) = (L_X \mathcal{T}) \otimes S + \mathcal{T} \otimes L_X(S)$

② If $Y_1, \dots, Y_n$ are any v.f's

$L_X \mathcal{T}(Y_1 \dots Y_n) =$

$$(L_X \mathcal{T})(Y_1 \dots Y_n) + \mathcal{T}(L_X Y_1, \dots, Y_n) + \dots + \mathcal{T}(Y_1, \dots, L_X Y_n).$$

These rules determine how L_X acts on all tensor fields (sections of $T(r,s)$).

Eg: If Y is v.f. then

$$L_X(Y(f)) = X(Y(f)) = (L_X Y)(f) + Y(X(f))$$

$$\Rightarrow L_X Y = [X, Y].$$