If $\mathcal{T}$ is section of $T(r,s)$, define

$$(L_X \mathcal{T})(p) = \frac{d}{dt}\Big|_{t=0} (\Phi_t^X)^* \mathcal{T} \Big|_p .$$

- More practically $L_X \mathcal{T}$ can be defined algebraically.

① $L_X(\mathcal{T} \otimes S) = (L_X \mathcal{T}) \otimes S + \mathcal{T} \otimes L_X(S)$

② If $Y_1, \dots, Y_n$ are any v.f's

$$L_X T(Y_1 \dots Y_n) =$$

$$(L_X T)(Y_1 \dots Y_n) + T(L_X Y_1, \dots, Y_n) + \dots + T(Y_1, \dots, L_X Y_n).$$

These rules determine how $L_X$ acts on all tensor fields (sections of $T(r,s)$).

Eg: If $Y$ is v.f. then

$$L_X(Y(f)) = X(Y(f)) = (L_X Y)(f) + Y(X(f))$$

$$\Rightarrow \quad L_X Y = [X, Y].$$

Eg: If $\alpha \in T(0,2) = T^*M \otimes T^*M$
then, in local coordinates

$$\alpha = \alpha_{ij}\, dx^i \otimes dx^j, \quad X = X^k \partial_k$$

$$L_X \alpha = (X\alpha_{ij})\, dx^i \otimes dx^j + \alpha_{ij}\left(L_X dx^i \otimes dx^j + dx^i \otimes L_X dx^j\right)$$

$$= \left(X^k \partial_k \alpha_{ij} + \partial_i X^k \alpha_{kj} + \partial_j X^k \alpha_{ik}\right) dx^i \otimes dx^j.$$

[We have used:

$$L_X dx^i = d\, i_X dx^i = d X^i = \partial_j X^i\, dx^j.]$$

If $\alpha$ is a symmetric tensor field of type $(0,2)$ (so $\alpha_{ij} = \alpha_{ji}$) then note that $L_X \alpha$ is also symmetric.

- Now, we return to the Riem. metric $g$ on $M$.

$$(-)^\flat : TM \xrightarrow[\cong]{g} T^*M$$

So every v.f. $X$ determines a one-form $X^\flat$.

$\rightsquigarrow \quad dX^\flat \in \Omega^2(M)$.

- We return to the covariant derivative on $M = \mathbb{R}^n$.

$$\nabla_Y X = (dX^i)(Y)\,\partial_i \,.$$

<u>Prop</u>: On $M = \mathbb{R}^n$, w/ $g = g_{std}$ have

$$2g(\nabla_Y X, Z) = (L_X g)(Y, Z) + (dX^\flat)(Y, Z).$$

for all v.f.'s $X, Y, Z$.

<u>Pf</u>: $(L_X g)(\partial_k, \partial_l) + (dX^\flat)(\partial_k, \partial_l)$

$$= (\cancel{X \cdot \delta_{kl}})^{0} - g(L_X \partial_k, \partial_l) - g(\partial_k, L_X \partial_l)$$

$$+ \partial_k g(X, \partial_l) - \partial_l g(X, \partial_k) - g(X, \cancel{[\partial_k, \partial_l]}^{0})$$

$$= -g\left(-(\partial_k X^i)\partial_i, \partial_l\right) - g\left(\partial_k, -(\partial_l X^j)\partial_j\right)$$
$$+ \partial_k X^l - \partial_l X^k.$$

$$= \partial_k X^l + \partial_l X^k + \partial_k X^l - \partial_l X^k$$

$$= 2\partial_k X^l.$$

On the other side:

$$2g(\nabla_{\partial_k} X, \partial_l) = 2g((\partial_k X^i)\partial_i, \partial_l)$$
$$= 2\partial_k X^l. \qquad \blacksquare$$

The key idea is that we can use this to define the covariant derivative on any Riem manifold.

- We have used, $\theta \in \Omega^1 \rightsquigarrow$

$$(d\theta)(X,Y) = X\cdot\theta(Y) - Y\cdot\theta(X) - \theta([X,Y]).$$

- Let $\nabla X$ be the $(1,1)$-tensor s.t.

$$\langle \nabla X, Y\rangle = \nabla_Y X.$$

Theorem : [The fundamental theorem of Riem geometry]

The linear map

$$\nabla : \Gamma(TM) \longrightarrow \Gamma(TM \otimes T^*M)$$

is the unique one s.t.

① $\nabla_Y (fX) = (Y \cdot f) X + f \nabla_Y X$

(Derivation)

② $\nabla_X Y - \nabla_Y X = [X, Y]$

(torsion-free).

③ $Z \cdot g(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$.

(preserves metric).

Pf: ① Note $L_{fX}\, g = f L_X\, g$

and $(fX)^\flat = f X^\flat$.