HOMEWORK 1 SOLUTIONS

Problem 1. In class we proved a formula for the symbol

$$\sigma_k(D)(x,\xi) = \lim_{t\to\infty} t^{-k}(e^{-\mathrm{i}tf}De^{\mathrm{i}tf})(x) \tag{1}$$

as endomorphisms of E_x. Here f is any smooth function such that $\mathrm{d}f(x) = \xi$.

Taking k t-derivatives of the quantity inside of the limit, we can write this as

$$\sigma_k(D)(x,\xi) = \frac{(-\mathrm{i})^k}{k!}(\mathrm{ad}\, f)^k D = \frac{(-\mathrm{i})^k}{k!}[\cdots[D,f],f]\cdots,f]. \tag{2}$$

In particular if H is a generalized Laplacian then

$$[[H,f],f] = -2\|\xi\|^2 = -2\|\mathrm{d}f\|^2. \tag{3}$$

Problem 2.

(a) There is a canonical section of $\mathrm{T}_M^* \otimes \mathrm{T}_M = \mathrm{End}(\mathrm{T}_M)$ given by the identity. This is the tautological one-form $\theta \in \Omega^1(M,\mathrm{T}_M)$.

(b) The equation $\mathrm{d}\theta + \omega\wedge\theta = 0$ follows from the fact that e_j are obtained via parallel transport of an orthonormal frame of $T_{x_0}M$.

(c) We first show that $(Eu, e_i) = \mathbf{x}^i$. Indeed, since the Levi–Civita connection is metric preserving

$$Eu(Eu,e_i) = (\nabla_{Eu}Eu, e_i) + (Eu, \nabla_{Eu}e_i) = (\nabla_{Eu}Eu, e_i). \tag{4}$$

To compute $\nabla_{Eu}Eu$ we recall that in normal coordinates the geodisics are precisely the rays $\mathbf{x}_t = t\mathbf{x}$. Since $Eu(\mathbf{x}_t) = t\dot{\mathbf{x}}_t$ we have

$$\nabla_{Eu}Eu = t\nabla_{\dot{\mathbf{x}}_t}(t\mathbf{x}_t) = t\dot{\mathbf{x}}_t = Eu. \tag{5}$$

Thus, $Eu(Eu,e_i) = (Eu,e_i)$. In other words, the function (Eu,e_i) is of homogenous degree one in normal coordinates.

In normal coordinates one has $e_i = \partial_i + O(\|\mathbf{x}\|)$, thus

$$(Eu,e_i) = \mathbf{x}^j(\partial_j,e_i) = \mathbf{x}^i + O(\|\mathbf{x}\|^2) = \mathbf{x}^i, \tag{6}$$

as desired.

(d) We have $L_{Eu}\theta = (\mathrm{d}i_{Eu} + i_{Eu}\mathrm{d})\theta = \mathrm{d}\mathbf{x} + i_{Eu}(-\omega\wedge\theta)$ where $\mathrm{d}\mathbf{x}$ denotes the vector one-form $(\mathrm{d}\mathbf{x}^1,\ldots,\mathrm{d}\mathbf{x}^n)$ and we have used $\mathrm{d}\theta + \omega\wedge\theta = 0$. Now $L_{Eu}\mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{x}$ and $L_{Eu}\mathbf{x} = \mathbf{x}$. Thus

$$(L_{Eu} - \mathbb{1})L_{Eu}\theta = -L_{Eu}i_{Eu}(\omega\wedge\theta). \tag{7}$$

But in this coordinate we have seen that $i_{Eu}\omega = 0$. Thus

$$(L_{Eu} - \mathbb{1})L_{Eu}\theta = \mathbf{x}L_{Eu}\omega = \mathbf{x}\iota_{Eu}\Omega. \tag{8}$$

This is the vector one-form $(\mathbf{x}^1\iota_{Eu}\Omega, \ldots, \mathbf{x}^n\iota_{Eu}\Omega)$.