The ABS construction

In this note we provide an overview of the construction of Atiyah, Bott, and Shapiro which provides, in part, a relationship between topological K-theory and Clifford modules. After a rapid introduction to K-theory we follow parts of the original reference [ABS64]. For a nice textbook review of K-theory see [Hat03].

1. A rapid introduction to K-theory

In this section the field $\mathbf{k}$ is either $\mathbf{R}$ or $\mathbf{C}$, and we consider $\mathbf{k}$-vector bundles on a space X. Write $\underline{\mathbf{k}}^n$ for the trivial bundle of rank n.

We introduce two equivalence relations on the set of vector bundles $\mathrm{Vect}(X)$.

- Stable isomorphism $\simeq_s$. Two vector bundles E_1, E_2 on X are *stably isomorphic*, we write $E_1 \simeq_s E_2$, if there exists an $N \geq 0$ and a bundle isomorphism
$$e_1 \oplus \underline{\mathbf{k}}^N \simeq E_2 \oplus \underline{\mathbf{k}}^N.$$
- Equivalence relation $\sim$. More generally, we say $E_1 \sim E_2$ if there exists $N, M \geq 0$ such that
$$E_1 \oplus \underline{\mathbf{k}}^N \simeq E_2 \oplus \underline{\mathbf{k}}^M.$$

Proposition 1.1. *If X is compact Hausdorff, then the set of $\sim$-equivalence classes of vector bundles over X forms an abelian group with respect to direct sum $\oplus$. This group is denoted $\widetilde{K}(X)$.*

This group $\widetilde{K}(X)$ is called the *reduced* K-group of X. The unreduced version is defined using the equivalence relation $\simeq$, except it is slightly more complicated. The issue is that only the class of the zero vector bundle is invertible in the set of $\simeq_s$-equivalence classes with respect to direct sum. Nevertheless, we can "cancel" bundles in some sense.

Lemma 1.2. *Suppose X is compact and E is a vector bundle on X. There exists a vector bundle E' on X such that $E \oplus E'$ is trivializable.*

PROOF. Suppose $\{U_i\}$ is a finite trivializing cover for E, so we have trivializations $\phi_i\colon U_i \times \mathbf{k}^n \to E$. There is a partition of unity $\{f_i\colon X \to [0,1]\}$ subordinate to this finite cover. For each i this allows us to define a vector bundle homomorphism

$$f_i \cdot \phi_i^{-1}\colon E \to X \times \mathbf{k}^n \tag{1}$$

Together, thus, we get a vector bundle homomorphism

$$\oplus f_i \cdot \phi_i^{-1}\colon E \to X \times \mathbf{k}^N \tag{2}$$

where N is n times the cardinality of the set parametrizing the cover. This morphism is fiberwise injective since f_i is non-vanishing at at least one point of U_i. So, we have embedded E into a trivial vector bundle. Fix an inner product on the trivial vector bundle (this exists by paracompactness). Then $E \oplus E^\perp \simeq \underline{\mathbf{k}}^{\oplus N}$ □

For example, if M has a framing and $S \subset M$ is a submanifold, then the sum of TS with the normal bundle $N_M S$ is trivializable.

From this lemma, we see that if $E_1 \oplus E_2 \simeq_s E_1 \oplus E_3$ then we can add $E_1^\perp$ to both sides to see that $E_2 \simeq_s E_3$. It follows that the set of $\simeq_s$-equivalence classes forms a semi-group with respect to $\oplus$. The K-group $K(X)$ is the group completion of this semi-group. (Think of the positive rational numbers $\mathbf{Q}_{>0}$ as the group completion of the semi-group of positive natural numbers $\mathbf{Z}_{>0}$ under multiplication.)

1.1. By definition, one represents elements of $K(X)$ as classes of formal differences

$$[E_1 - E_2]$$

where E_1, E_2 are bundles on X. Then $[E_1 - E_2] = [E_1' - E_2']$ if and only if there is a stable equivalence

$$E_1 \oplus E_2' \simeq_s E_1' \oplus E_2.$$

The group operation is the obvious thing

$$[E_1 - E_2] + [E_2 - E_2'] = [(E_1 \oplus E_2) - (E_1' \oplus E_2')].$$

Note that every element in $K(X)$ can be represented by a formal difference $[E - \underline{\mathbf{k}}^n]$ for some bundle E and some integer n.

There is a natural homomorphism $K(X) \to \widetilde{K}(X)$ defined by $[E - \underline{\mathbf{k}}^n] \to [E]$ whose kernel consists of classes of the form $\underline{\mathbf{k}}^0 - \underline{\mathbf{k}}^n$. Hence, $E \simeq_s \underline{\mathbf{k}}^m$ for some m. Thus, the kernel of this homomorphism is $\mathbf{Z}$ and there is an isomorphism $K(X) \simeq \widetilde{K}(X) \oplus \mathbf{Z}$ coming from a splitting of this homomorphism $K(X) \to K(pt)$ whose kernel is exactly

$\widetilde{K}(X)$. The subgroup $\widetilde{K}(X)$ of $K(X)$ is an ideal and hence a ring in its own right with respect to tensor product.

A map $f\colon X \to Y$ determines a ring map on K-theory $f^*\colon K(Y) \to K(X)$ which sends a vector bundle $[E]$ on Y to the vector bundle $[f^*E]$ on X. Likewise, reduced K-theory is also functorial.

One of the main results about K-theory is Bott periodicity. It is easiest to state for complex K-theory, so for now we work over $\mathbf{C}$.

From now on $K(X)$ and $\widetilde{K}(X)$ will denote complex K-theory and reduced complex K-theory. Real K-theory is denoted $KO(X)$ and its reduced version is $\widetilde{KO}(X)$.

Let $L = \mathcal{O}(-1)$ be the tautological line bundle on $\mathbb{P}^1 = S^2$ (this is a holomorphic line bundle, but K-theory only knows its structure as a complex line bundle).

Lemma 1.3. *There is a bundle isomorphism $L \otimes L \oplus \underline{\mathbf{C}} \simeq L \oplus L$.*

PROOF. On $\mathbb{P}^1$ the data of a vector bundle is specified the homotopy class of a map

$$S^1 \to GL(2, \mathbf{C}) \tag{3}$$

Let E_t be a continuous path in $GL(2, \mathbf{C})$ which satisfies

$$E_0 = \mathbb{1}, \quad E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

Such a path exists by connectedness. Consider the path in $Map(S^1, GL(2, \mathbf{C}))$:

$$f_t(z) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} E_t \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} E_t. \tag{5}$$

This is a path from the clutching function for $L \oplus L$ to the clutching function for $L \otimes L \oplus \underline{\mathbf{C}}$. □

It follows that there is a ring homomorphism

$$\mathbf{Z}[L]/(L-1)^2 \to K(S^2). \tag{6}$$

THEOREM 1.4. *This is an isomorphism.*

If $a \in K(X)$ and $b \in K(Y)$ then define $a \star b = p_1^*(a) \otimes p_2^*(b) \in K(X \times Y)$. This is called the external product. Without much more work, one can show that

$$L \star (-)\colon \mathbf{Z}[L]/(L-1)^2 \otimes K(X) \to K(S^2 \times X) \tag{7}$$

is an isomorphism of rings.

The reduced version of this homomorphism is of the form

$$\widetilde{K}(X) \to \widetilde{K}(S^2X) \tag{8}$$

and explicitly sends $x \mapsto (L-1) \star x$.

THEOREM 1.5 (Bott periodicity). *This is an isomorphism of rings.*

1.2. We now construct a graded version of K-theory.

The suspension SX of a space X is defined to be the quotient of the cylinder $X \times [0,1]$ where we identify $X \times \{0\}$ to a single point and $X \times \{1\}$ to a single point. For example S^{n+1} is homeomorphic to $S(S^n)$. Equivalently, S^n can be seen as the nth fold suspension of $S^0 = \{-1, +1\}$.

The graded version of K-theory is defined using the suspension. Define, for reduced K-theory

$$\widetilde{K}^{-n}(X) \stackrel{\text{def}}{=} \widetilde{K}(S^nX). \tag{9}$$

In total, define the graded abelian group

$$\widetilde{K}^{-\bullet}(X) \stackrel{\text{def}}{=} \oplus_{n\geq 0}\widetilde{K}^{-n}(X). \tag{10}$$

The negative grading is chosen to match with with cohomological grading. For non-reduced, one defines $K^{-n}(X) = \widetilde{K}^{-n}(X_+)$.

For $A \subset X$ a closed subspace, there is the following exact sequence in reduced K-groups

$$\cdots \to \widetilde{K}(SX) \to \widetilde{K}(SA) \to \widetilde{K}(X/A) \to \widetilde{K}(X) \to \widetilde{K}(A) \tag{11}$$

The right most map is restriction of vector bundles from X to A, and the second to right most map is pulling back along the quotient map $X \to X/A$.

We use this exact sequence to find a relationship of K-theory with products of spaces. Let $X \wedge Y \stackrel{\text{def}}{=} X \times Y / X \vee Y$, where $X \vee Y$ is defined using specified basepoints of X and Y. Then we can apply the above long exact sequence to the pair $(X \times Y, X \vee Y)$ deduce an isomorphism

$$\widetilde{K}(X \times Y) \simeq K(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y). \tag{12}$$

We thus get a different sort of external product, defined by the composition

$$\widetilde{K}(X) \otimes \widetilde{K}(Y) \xrightarrow{\star} \widetilde{K}(X \times Y) \to \widetilde{K}(X \wedge Y) \tag{13}$$

where the last map is projection. Replacing X, Y by $S^i X, S^j Y$ defines a product

$$\widetilde{K}^i(X) \otimes \widetilde{K}^j(Y) \to \widetilde{K}^{i+j}(X \wedge Y). \tag{14}$$

Finally, in the case that $X = Y$ we can additionally compose with the restriction along the diagonal map $X \to X \wedge X$ to get a product

$$\widetilde{K}^i(X) \otimes \widetilde{K}^j(X) \to K^{i+j}(X). \tag{15}$$

Proposition 1.6. *This endows $\widetilde{K}^\bullet(X)$ with the structure of a commutative graded ring.*

From Bott periodicity, one immediately obtains a graded ring isomorphism

$$K^\bullet(\star) \simeq \mathbf{Z}[L] \tag{16}$$

where $L \in \widetilde{K}^{-2}(\star) = \widetilde{K}(S^2)$ is a degree -2 generator and represents the canonical line bundle L.

Bibliography

[ABS64] M. F. Atiyah, R. Bott, and A. Shapiro. "Clifford modules". *Topology* 3.suppl (1964), pp. 3–38. URL: https://doi.org/10.1016/0040-9383(64)90003-5.

[Hat03] A. Hatcher. *Vector Bundles and K-Theory*. http://www.math.cornell.edu/~hatcher. 2003.