April 10 $\quad (M, g)$ oriented, Riemannian manifold.

Recall we have defined the twistor space

$$\tau(M) = Fr_M^{SO} \times_{SO(2n)} \mathbb{P}(PS^+)$$

$$\cong Fr_M^{SO} \times_{SO(2n)} \left(SO(2n)/U(n) \right)$$

$$\cong Fr_M^{SO} / U(n).$$

- Fix an almost cpx str J_0 of $\mathbb{R}^{2n}$. Then any $A \in so(2n) \cong \Lambda^2 \mathbb{R}^{2n}$ can be written

$$A = \frac{1}{2}\left(A - J_0 A J_0\right) + \frac{1}{2}\left(A + J_0 A J_0\right)$$

$$\Rightarrow \quad so(2n) = u(n) \oplus m$$

where

$$m = \left\{ A \in so(2n) \;\middle|\; A J_0 = -J_0 A \right\}$$

Thus $T_{J_0}\left(SO(2n / U(n)\right) \cong m.$

$$\mathcal{J} : A \longmapsto J_0 A$$

endows $\mathfrak{m}$, and hence $SO(2n)/U(n)$ w/ an almost cplx structure.

$$\mathfrak{m}_{\mathbb{C}} = \mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{m}_+ \oplus \mathfrak{m}_-$$

where J_0 acts on $\mathfrak{m}_\pm$ as $\pm i$.

$$\mathfrak{m}_\pm = J^\pm \mathfrak{m}, \quad J^\pm = \frac{1}{2}\left(1 \mp i J_0\right).$$

- The Levi-Civil connection determines a decomposition

$$T\tau(M) = \mathcal{H} \oplus \mathcal{V}$$

where $\mathcal{H} = \pi^* TM$ and $\mathcal{V}$ = vertical tangent bundle.

Have constructed an almost cplx str. on $\mathcal{V}$.
Since $\mathcal{H}_z \cong T_{\pi(z)} M$, there is also a tautological almost cplx str. on $\mathcal{H}$.

- Weyl tensors : Let (M, g) be Riem. manifold, ∇ is the L.C. connection.

Then $R = F_\nabla \in \Omega^2(M, \mathrm{End}(T_M))$.

Symmetries : So, a priori, the Riemann curvature is a section

$$R \in \Gamma(M, T_M^* \otimes T_M^* \otimes T_M^* \otimes T_M)$$
$$\cong$$
$$\Gamma(M, (T_M^*)^{\otimes 4}).$$
$$\cup$$

In fact:
$$R \in \Gamma(M, S^2(\Lambda^2 T_M^*)).$$

Locally, denote $V = T_x M$, so we are looking at

$$S^2(\Lambda^2 V).$$

The Bianchi identity further constrains the symmetry:

$$b : S^2(\Lambda^2 V) \longrightarrow \Lambda^4 V$$

$\rightsquigarrow$

$$R_x \in \ker b \subset S^2(\Lambda^2 V).$$

· Ricci contraction $r : \ker b \longrightarrow S^2 V$

Thm: When $\overset{n}{\overset{\shortparallel}{\dim M}} \geq 4$ have

$$\ker b = \mathbb{R} \oplus S^2_0 V \oplus \ker r.$$

$\circlearrowleft$

$O(n)$

(Ricci scalar, Ricci trace-free, Weyl).

Thm: The Weyl tensor is conformally invariant. Moreover if $n \geq 4$ then M is conformally flat $\Longleftrightarrow$ $W \equiv 0$.

When $\dim M = 4$ we have

$$\Lambda^2 T^*_M = \Lambda^2_+ \oplus \Lambda^2_- .$$

This induces a conformally invt decomposition

$$W = W_+ + W_- .$$

<u>Ex</u>: $\dim M = 4k$ compact oriented. Then $H^{2k}(M, \mathbb{R})$ is a symmetric vector space.

$\rightsquigarrow$ symmetric bilinear form

$$B : H^{2k} \times H^{2k} \longrightarrow H^{4k} \cong \mathbb{R}$$

$$\rightsquigarrow \quad Q(x) = B(x, x) .$$

$$\sigma(M) = \text{signature of } Q$$

<u>Thm</u>: $\dim M = 4$:

$$\sigma(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) dvol .$$

Rmk: When $\dim M = 4$ and M is Kähler then W_+ is completely determined by the Ricci scalar curvature.

Thm: [Atiyah - Hitchin - Singer] M oriented, Riem. manifold of $\dim_{\mathbb{R}} = 4$.

- Then the almost cplx str. of $\mathcal{T}(M)$ is integrable $\Leftrightarrow$ $W_+ \equiv 0$.

- This cplx str. is Kähler $\Leftrightarrow$ M is conformally equivalent to S^4 or $\mathbb{C}P^2$.

[When $\dim > 4$ the only Kähler twistor spaces arise from even spheres $M = S^{2n}$].

Ex: • $M = \mathbb{R}^4$. We will see how it is natural to view

$$\mathcal{T}(M) \cong \mathrm{Tot}\left(\begin{array}{c} \mathcal{O}(1) \oplus \mathcal{O}(1) \\ \downarrow \\ \mathbb{CP}^1 \end{array} \right)$$

• $M = S^4$. $\mathcal{T}(S^4) \cong \mathbb{CP}^3$.

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• Let's turn to the question of integrability for sections of $\mathcal{T}(M)$.

A "pure spinor field" is a section of $\mathcal{T}(M)$ $\longleftrightarrow$ almost cplx str. of M cptble w/ metric + or..

Prop: A pure spinor field σ determines an integrable cplx str. $\Leftrightarrow$

$$v \cdot \nabla_w \sigma - w \cdot \nabla_v \sigma = 0$$

$$\forall \; v, w \in T(T^{0,1}_M).$$

Pf : $T^{0,1}_M$ is defined in terms of the pure spinor field σ by

$$T^{0,1}_M = \left\{ v \in \Gamma(T_M \otimes \mathbb{C}) \;\middle|\; v \cdot \sigma = 0 \right\}.$$

Take $w \cdot \sigma = 0$ and apply ∇_v :

$$0 = \nabla_v (w \cdot \sigma) = \nabla_v w \cdot \sigma + w \cdot \nabla_v \sigma .$$

Similarly $0 = \nabla_w v + v \cdot \nabla_w \sigma$.

$\Rightarrow$
$$0 = \underbrace{\left(\nabla_v w - \nabla_w v\right)}_{\nabla_{[v,w]}\sigma} \sigma + v \cdot \nabla_w \sigma - w \cdot \nabla_v \sigma .$$

So $v \cdot \nabla_w \sigma - w_v \nabla_w \sigma = 0$

$\Leftrightarrow [v,w] \cdot \sigma = 0 \Leftrightarrow [v,w] \in \Gamma(T^{0,1}_M)$.

∎

Thm: Let M be or. Riem. of dim $= 2m$.
Then the almost cplx str. determined by

$$s \in \Gamma(\tau(X))$$

is integrable $\Leftrightarrow$ s is (almost) holomorphic.

Pf: Pick $[\sigma] = s$. Then s is holomorphic

$$\Leftrightarrow_{(**)} \quad \nabla_v \sigma = \lambda_v \sigma \qquad \forall\, v \in \Gamma(T^{0,1})$$

where λ_v is some function depending on v.

Our prop $\Rightarrow$

$$s = \bigcap_i \ker(\mu_{\bar{\varepsilon}_i}) \quad , \{\bar{\varepsilon}_i\} \text{ hermitian frame for } T^{0,1}$$

Since $\bar{\varepsilon}_j^2 = 0$, we can use the above prop to see that:

$$-\bar{\varepsilon}_i \bar{\varepsilon}_j \nabla_{\bar{\varepsilon}_j} \sigma = \bar{\varepsilon}_j \bar{\varepsilon}_i \nabla_{\bar{\varepsilon}_j} \sigma = \bar{\varepsilon}_j \bar{\varepsilon}_j \nabla_{\bar{\varepsilon}_i} \sigma = 0$$

$\Rightarrow$

$$\bar{\varepsilon}_j \nabla_{\bar{\varepsilon}_i} \sigma \in s \subset S^+ \quad \forall\, i, j.$$

But also $\bar{\varepsilon}_j \nabla_{\bar{\varepsilon}_i} \sigma \in S^-$ by type reasons.

$\Rightarrow$

$$(*) \qquad \bar{\varepsilon}_j \nabla_{\bar{\varepsilon}_i} \sigma = 0 \quad \forall j.$$

Now: Consider the symmetric $C^\infty(X)$-bilinear

form $\beta(v, w) = w \nabla_v \sigma$, $\quad v, w \in \Gamma(T^{0,1}_M)$

$(*) \Rightarrow \beta = 0$. So $\nabla_v \sigma \in s = [\sigma]$ for

all $v \in \Gamma(T^{0,1})$.

Conversely, if $\nabla_v \sigma = \lambda_v \sigma$ then

automatically $w \cdot \nabla_v \sigma = 0$. $\qquad \blacksquare$