April 15

Yang-Mills Equations : Let (M, g) be a Riemannian 4-manifold. Let G be a compact Lie group (eg $G = U(1)$ or $SU(2)$ work just fine.)

The Yang-Mills eqns are a system of PDE's which can be expressed in terms of connections on a principal G-bundle on M.

- We start w/ the trivial G-bundle on M.

A connection for the trivial bundle is of the form

$$\nabla = d + A \quad , \quad A \in \Omega^1(M, \mathfrak{g}).$$

The curvature is $F^{\nabla} = dA + \frac{1}{2}[A, A] \in \Omega^2(M, \mathfrak{g})$.

- The group of gauge transformations acts on the space of connections: $g \in C^{\infty}(M; G)$

$$A \xmapsto{g} g^{-1} A g + g^{-1} dg .$$

$$\rightsquigarrow \qquad F^\nabla \xmapsto{g} g^{-1} F^\nabla g$$

The Bianchi identity asserts that F^∇ is a closed 2-form valued in $\mathrm{End}(E)$:

$$dF^\nabla = 0.$$

This holds for any connection ∇.

The metric has appeared!

The Yang-Mills eqn is $\boxed{d * F = 0}$

<u>Lemma</u>: 1) The YM eqns are gauge invariant.

2) The YM eqn are conformally invariant.

<u>Pf</u>: 1) $d * F \xmapsto{g} d*(g^{-1} F g)$.

$$= g^{-1}(d*F)\, g = 0.$$

2) Sps $g \rightsquigarrow \lambda g$, $\lambda : M \to \mathbb{R}_+$.

Locally we have o.n.b. $\{e_1, \dots, e_4\}$

The new o.n.b is $\left\{ \frac{e_1}{\sqrt{\lambda}}, \dots, \frac{e_4}{\sqrt{\lambda}} \right.$

So, if we have a two-form

$$e_i \wedge e_j \overset{\lambda}{\longmapsto} \frac{1}{\lambda} e_i \wedge e_j .$$

But

$$* e_i \wedge e_j = \varepsilon_{ijkl} e_k \wedge e_l$$
$$\underset{\lambda}{\longmapsto} \frac{1}{\lambda} \varepsilon_{ijkl} e_k \wedge e_l .$$

So the $*$ operator is conformally invariant when acting on 2-forms. $\boxed{///}$

- There are other easier sorts of conformally invariant eqn's.

(i) Laplace's eqn. $\Delta \varphi = 0$
where $\varphi \in C^\infty(M)$.

(2) Dirac equation $\not{\partial}\psi = 0$

spinor bundle

where $\psi \in \Gamma(M, S_+)$

Riem. spin manifold.

$$\not{\partial} : \Gamma(M, S_+) \xrightarrow{\nabla_{L.C.}} \Gamma(M, T^*_M \otimes S_+) \longrightarrow \Gamma(M, S_-).$$

$\parallel$

Dirac operator / We move on to the classification of conformal PDE's.

- Jets : Let E be a vector bundle on X, any manifold. Define :

$$J^k_p(E) = \left\{ s \in \Gamma(U, E) \;\middle|\; U \text{ nbd of } p \in X \right\} \Big/ \sim$$

where $s \sim s' \iff \left.\dfrac{\partial^k s}{\partial x^I}\right|_p = \dfrac{\partial^k s'}{\partial x^I}_p$

for all $I = (i_1, \dots, i_k)$ multi-index.

These combine to form the k-th jet bundle

$$\begin{array}{c} J^k E \\ \downarrow \\ X \end{array}$$

There is an exact sequence of vector bundles

$$0 \to S^k T^*_X \otimes E \to J^k E \to J^{k-1} E \to 0$$

(Think of the sequence

$$\begin{matrix} \left\{\begin{matrix}\text{homogeneous}\\ \text{degree } k\\ \text{polynomials}\end{matrix}\right\} \to \mathbb{C}[z_1, \ldots, z_n] / I^{k+1} \to \mathbb{C}[z_1, \ldots, z_n] / I^k \to 0 \\ \| \\ 0 \to S^k \mathbb{C} \end{matrix}$$

.)

- Sps E has a connection ∇.

$$\nabla : \Gamma(E) \to \Gamma(T^*_X \otimes E).$$

Differential operator

Look at

bundle homo

$$\nabla$$

$$0 \to T^*_X \otimes E \to J^1 E \to E \to 0$$

<u>Lem</u>: ∇ determines a splitting.

<u>Pf</u>: A general fact (definition) is that a differential operator $\mathcal{D} : \Gamma(E) \to \Gamma(F)$

is of order k $\iff$ it extends to a

vector bundle homomorphism $J^k E \longrightarrow F$.

So ∇ is first-order thus it extends to a bundle homomphism $\nabla : J^1 E \longrightarrow T^{*}_X \otimes E$. $\blacksquare$

- Conformal structures: Let $(X, [g])$ be a conformal structure.

$\iff$ Reduction of structure group of $Fr_X^{GL_n}$

$$\text{to } CO(n) = \{ aA \mid a \in \mathbb{R}_+ ,\ A \in SO(n) \}$$

- An irrep of $CO(n)$ is determined by an irrep of $SO(n)$ plus a weight $w \in \mathbb{R}$.

Let

$$CE(n) = CO(n) \ltimes \mathbb{R}^n \circlearrowleft \text{translation}.$$

$$\cong$$

$$\left\{ \begin{array}{c} \text{group of automorphisms} \\ \text{of 1-jet of conf} \\ \text{structure at a pt} \end{array} \right\}.$$

Sps $\quad CO(n) \underset{\rho}{\circlearrowright} \mathbb{E}$. Let's define ass. bundle

$$E \overset{def}{=} Fr_X^{CO(n)} \times_\rho \mathbb{E}.$$

<u>Prop</u>: The vector bundle $J^1 E$ admits a reduction of structure to the group $CE(n)$.

Explicitly, if we use the L.C. connection to get a splitting

$$J^1 E = E \oplus E \otimes T_X^*$$

then translations $x \in \mathbb{R}^n$ act by

$$v \overset{x}{\longmapsto} \sum_i \rho(x \otimes e_i^* - e_i \otimes x^*) v \otimes e_i + w\, v \otimes x.$$

where:

- $v \in \mathbb{E}$
- $\{e_i\}$ o.n.b for $T_p^* X$.
- $x \in \mathbb{R}^n$
- w is the conformal wt of the representation.

- A subspace $V_x \subset J^1 E_x$ which is invariant under $CE(n)$ defines a conformally invariant differential equation as $x \in X$ varies.

If $\rho \curvearrowright SO(n)$ $\mathbb{E}$ is an irrep of $SO(n)$, then any $CE(n)$ invariant submodule of

$$J^1 E_x \cong \mathbb{E} \oplus \mathbb{E} \otimes \mathbb{R}^n$$

which projects onto $\mathbb{E}$ is of the form

$$\mathbb{E} \oplus \operatorname{Im}(B + w \mathbb{1})$$

where $B : \mathbb{E} \otimes \mathbb{R}^n \longrightarrow \mathbb{E} \otimes \mathbb{R}^n$ is

$$B(v \otimes x) = \sum_i \rho(x \otimes e_i - e_i \otimes x)\, v \otimes e_i .$$

$\Rightarrow$ get proper invt subspace when $-w$ = eigenvalue of B. The corresponding differential operator sends sections of E to sections of the bundle associated to $\ker(B + w\mathbb{1})$.

- Since B is $so(n)$-inv't, we can express it in terms of Casimirs:

$$\frac{1}{2} B = C(E) \otimes \mathbb{1} + \mathbb{1} \otimes C(\mathbb{R}^n) - C(E \otimes \mathbb{R}^n).$$

where

$$C(E) = \sum_a \rho(X_a)$$

where $\{X_a\}$ is onb for $so(n)$ w/ respect to the Killing form

$$\kappa(X,Y) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$$

Trace taken in the adjoint representation.

<u>Claim</u>: For $\mathfrak{g} = so(n)$, $\kappa(X,Y) = (n-2)\,\mathrm{tr}(XY)$

trace in defining rep^n.

We specialize to $n=4$ dimensions. Recall the complex spin reps $S_\pm$ of

$$\mathrm{Spin}(4) \simeq SU(2) \times SU(2).$$

$$\dim_{\mathbb{C}} S^{\pm} = 2, \quad C(S^{\pm}) = -\frac{3}{8}.$$

More generally, we consider

- $E = S^m S_+ \otimes S^n S_-$ which is of dimension $(m+1)(n+1)$.

  and $C(E) = -\frac{m(m+2)}{8} - \frac{n(n+2)}{8}$.

In particular

$$\mathbb{R}^4 \cong S_+ \otimes S_- \text{ has } C(\mathbb{R}^n) = -2 \cdot \frac{3}{8} = -3/4.$$

Ex: ① $\mathbb{E}$ = triv 1-dim'l rep. Then $w = 0$ and there is a unique conformally inv't differential operator

$$d : C^\infty(M) \longrightarrow \Omega^1(M).$$

② $\mathbb{E} = \mathbb{R}^4$ = defining rep. Recall $\mathbb{R}^4 = S_+ \otimes S_-$.

$\Rightarrow$

$$\mathbb{R}^4 \otimes \mathbb{R}^4 = S_+^2 \otimes S_-^2 \oplus S_+^2 \oplus S_-^2 \oplus \mathbb{1}.$$

Casimir: $(-2, -1, -1, 0)$.

Since $C(\mathbb{R}^n) = -3/4$, we see:

- $S_+^2 \otimes S_-^2$ need $w = -2 \cdot \left(-2 \cdot \frac{3}{4} + 2\right)$

$$= -2\left(-3/2 + 2\right) = -2\left(1/2\right) = -1.$$

This is the operator:

$$\begin{array}{ccc} \Gamma(TM) & \xrightarrow{L_g} & \Gamma(S_+^2 \otimes S_-^2) \\ \| & & \| \\ \mathrm{Vect}(M) & \xrightarrow{L_g} & \Gamma(S_0^2 T_M^*) \end{array}.$$

$$\ker(L_g) = \left\{ \begin{array}{c} \text{conformal} \\ \text{vector fields} \end{array} \right\}.$$

- S^2_+, need $w = -2\left(-\frac{3}{2}+1\right) = 1$.

This is

$$\begin{tikzcd} \Gamma(T^*_M) \arrow[r] \arrow[d, equal] & \Gamma(S^2_+) \arrow[d, equal] \\ \Omega^1 \arrow[r, "d_+ = d - d^*"'] & \Omega^2_+ \end{tikzcd}$$

- S^2_-, need $w = 1$. This is

$$\begin{tikzcd} \Gamma(T^*_M) \arrow[r] \arrow[d, equal] & \Gamma(S^2_-) \arrow[d, equal] \\ \Omega^1 \arrow[r, "d_- = d + d^*"'] & \Omega^2_- \end{tikzcd}$$

- 1, need $w = 3$. This is:

$$\Omega^3 \xrightarrow{d} \Omega^4.$$