Clifford algebra

1. Basic definitions

In this note we introduce the rudiments of Clifford algebra. For more details we refer to [LM89, Chapter I].

1.1. Let $\mathbf{k}$ be a field (with characteristic different from 2) and let V be a $\mathbf{k}$-vector space. A quadratic form is a symmetric, bilinear map $\langle -,- \rangle \colon V \times V \to \mathbf{k}$ which is non-degenerate in the sense that $\langle v, w \rangle = 0$ for all $w \in V$ implies $v = 0$. We will write $q(v) = \langle v, v \rangle$ in what follows.

The **Clifford algebra** $\mathrm{C}\ell(V, q)$ associated to V is the quotient of the tensor algebra

$$T(V) = \oplus_{k \geq 0} V^{\otimes k} \tag{1}$$

by the two-sided ideal generated by elements

$$v \otimes v - q(v) \cdot 1 \,, \quad v \in V. \tag{2}$$

Notice that the relation $v^2 = -q(v)1$, for all $v \in V$, can be equivalently written as the relation

$$v \cdot w + w \cdot v = -2\langle v, w \rangle 1 \tag{3}$$

for all $v, w \in V$.

Proposition 1.1. *The Clifford algebra* $\mathrm{C}\ell(V, q)$ *is the universal algebra for which:*

- *there is an injection* $i \colon V \hookrightarrow \mathrm{C}\ell(V, q)$*,*
- *let* $\phi \colon V \to A$ *be a linear map of* V *into a (unital)* $\mathbf{k}$*-algebra* A *such that*

$$\phi(x)^2 = -q(x)1.$$

Then there exists a unique homomorphism $\widetilde{\phi} \colon \mathrm{C}\ell(V, q) \to A$ *such that* $\widetilde{\phi} \circ i = \phi$*.*

The orthogonal group $O(V, q)$ are the linear automorphisms of V which preserve q. If $f \colon V \to V$ is such an automorphism then, $f(v)^2 = -q(f(v))\mathbb{1} = q(v)\mathbb{1}$ in $\mathrm{C}\ell(V, q)$

for all $v \in V$. Thus f defines a unique algebra homomorphism $\widetilde{f}\colon \mathrm{C}\ell(V,q) \to \mathrm{C}\ell(V,q)$ with the property that $\widetilde{f}|_V = f$. Moreover, since f is bijective so is $\widetilde{f}$. Thus we have constructed a group embedding

$$O(V,q) \hookrightarrow \operatorname{Aut} \mathrm{C}\ell(V,q). \tag{4}$$

In fact, the group lies within the subgroup of *inner* automorphisms.

1.2. A *filtered* algebra is an algebra A with an exhaustive sequence of subspaces

$$0 = F^{-1}A \subset F^0 A \subset \cdots \subset F^\ell A \subset \cdots \subset F^\infty A = A$$

such that if $a \in F^p A, b \in F^q A$ then $ab \in F^{p+q} A$.

Let $A = \{F^\bullet A\}$ be a filtered algebra. The associated graded $\operatorname{gr} A$ has underlying vector space

$$\operatorname{gr} A = \oplus_{k \geq 0} F^k A / F^{k-1} A.$$

The product on A defines a product on $\operatorname{gr} A$, giving $\operatorname{gr} A$ the structure of an algebra for which the canonical map $A \to \operatorname{gr} A$ is a homomorphism.

There is a filtration on the tensor algebra $T(V)$ defined by

$$F^p T(V) \stackrel{\text{def}}{=} \oplus_{k \leq p} F^k V. \tag{5}$$

This induces a filtration on the Clifford algebra $\mathrm{C}\ell(V,q)$ such that

$$\operatorname{gr} \mathrm{C}\ell(V,q) \cong \wedge V, \tag{6}$$

where $\wedge V$ is the exterior algebra on V. This implies the following.

Lemma 1.2. *Suppose $\{e_i\}$ is a basis for V. Then*

$$e_{i_1} \cdots e_{i_k},$$

where $i_1 < \cdots < i_k$, $k \geq 0$, form a basis for $\mathrm{C}\ell(V,q)$. In particular, $\dim \mathrm{C}\ell(V,q) = 2^{\dim V}$.

1.3. Let $\mathrm{C}\ell^{ev,odd}(V,q)$ be the images of

$$\oplus_{i \geq 0} V^{\otimes 2i}, \quad \oplus_{i \geq 0} V^{\otimes 2i+1}$$

in $\mathrm{C}\ell(V,q)$, respectively.

Proposition 1.3. *Both $\mathrm{C}\ell^{ev,odd}(V,q)$ are subalgebras of $\mathrm{C}\ell(V,q)$ and*

$$\mathrm{C}\ell(V,q) = \mathrm{C}\ell^{ev}(V,q) \oplus \mathrm{C}\ell^{odd}(V,q). \tag{7}$$

Furthermore, the product decomposes as

$$\begin{aligned}
C\ell^{ev}(V,q) \times C\ell^{ev}(V,q) &\to C\ell^{ev}(V,q)\\
C\ell^{ev}(V,q) \times C\ell^{odd}(V,q) &\to C\ell^{odd}(V,q)\\
C\ell^{odd}(V,q) \times C\ell^{ev}(V,q) &\to C\ell^{odd}(V,q)\\
C\ell^{odd}(V,q) \times C\ell^{odd}(V,q) &\to C\ell^{ev}(V,q).
\end{aligned}$$

The axioms of the above proposition characterize what is called a $\mathbf{Z}/2$ graded algebra, or *superalgebra*. Notice that $\wedge V$ is naturally a $\mathbf{Z}/2$ graded algebra, and the canonical homomorphism $C\ell(V,q) \to \wedge V$ preserves the $\mathbf{Z}/2$ gradings.

Proposition 1.4. *There is a natural isomorphism*

$$C\ell(V_1 \oplus V_2, q_1 \oplus q_2) \cong C\ell(V_1,q_1) \otimes^{gr} C\ell(V_2,q_2) \tag{8}$$

where the tensor product is the ***graded*** *tensor product (see below) of* $\mathbf{Z}/2$ *graded algebras.*

The graded tensor product $A \otimes^{gr} B$ of $\mathbf{Z}/2$ graded algebras A, B differs from the usual tensor product of plain ungraded algebras. As a vector space, it does agree with the standard tensor product

$$A \otimes^{gr} B = A^{ev} \otimes B^{ev} \oplus A^{ev} \otimes B^{odd} \oplus A^{odd} \otimes B^{ev} \oplus A^{odd} \oplus B^{odd}.$$

The product, on the other hand, is defined by

$$(a \otimes x) \cdot (y \otimes b) = (-1)^{|x||y|} ay \otimes xb$$

where $a, y \in A$ and $b, x \in B$.

1.4. Here is an alternative description of the $\mathbf{Z}/2$ grading. Let $i_q \colon V \hookrightarrow C\ell(V,q)$ denote the canonical morphism. Consider the automorphism

$$\alpha \colon C\ell(V,q) \to C\ell(V,q) \tag{9}$$

which extends the linear map $v \mapsto -v$. Since $\alpha^2 = \mathbb{1}_{C\ell(V,q)}$ we have a decomposition

$$C\ell^{ev,odd}(V,q) = \{x \in C\ell(V,q) \mid \alpha(x) = (-1)^{ev,odd} x\}. \tag{10}$$

These are exactly the even/odd subspaces from above.

1.5. As another example of an involution consider the reversal of order map

$$v_1 \otimes \cdots \otimes v_k \mapsto v_k \otimes \cdots \otimes v_1. \tag{11}$$

This preserves the defining ideal so descends to a linear automorphism of the Clifford algebra. This automorphism is not compatible with the algebra structure in the usual sense. It is an anti-automorphism in the sense that $(\varphi\psi)^t = \psi^t \varphi^t$.

2. Pin and spin groups

2.1. Given any algebra A we let $A^\times \subset A$ denote the group of units; the group of elements which admit a multiplicative inverse. There is a group homomorphism

$$\mathrm{Ad}\colon A^\times \hookrightarrow \mathrm{Aut}\, A, \tag{12}$$

defined by $\mathrm{Ad}_a\colon x \mapsto axa^{-1}$.

In the case of the Clifford algebra $A = \mathrm{C}\ell(V,q)$ (and $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$) the group of units $\mathrm{C}\ell(V,q)^\times$ is a Lie group of dimension 2^n. The following is a useful computation.

Proposition 2.1. *Suppose $v \in V$ satisfies $q(v) \neq 0$. Then*

$$-\mathrm{Ad}_v(x) = x - 2\frac{\langle v, x\rangle}{\langle v, v\rangle} v. \tag{13}$$

The Lie algebra $\mathrm{Lie}\,\mathrm{C}\ell(V,q)$ is isomorphic to $\mathrm{C}\ell(V,q)$ as a vector space and the bracket is the commutator

$$[x,y] \stackrel{\text{def}}{=} xy - yx. \tag{14}$$

(In fact, any algebra A defines a Lie algebra by the commutator.) The derivative of the group-level adjoint defines a Lie algebra homomorphism

$$\mathrm{ad}\colon\ \mathrm{Lie}\,\mathrm{C}\ell(V,q) \to \mathrm{Der}\,\mathrm{C}\ell(V,q), \tag{15}$$

given by $\mathrm{ad}_y(x) = [y,x]$.

2.2. The orthogonal group $O(V,q) \subset GL(V)$ is the subgroup of linear isomorphisms $A\colon V \to V$ which preserve the bilinear form $q(Av) = q(v)$. An easy calculation implies that if $A \in O(V,q)$ then $\det A = \pm 1$. The subgroup $SO(V,q) \subset O(V,q)$ consists of elements with $\det A = 1$. This subgroup is connected.

The Lie algebra of $SO(V,q)$ is the Lie algebra of skew-symmetric matrices

$$\mathfrak{so}(V) = \{A\colon V \to V \mid \langle Av, w\rangle = -\langle v, Aw\rangle\}. \tag{16}$$

Proposition 2.2. *The map*

$$T\colon \wedge^2 V \to \mathfrak{so}(V) \tag{17}$$

which sends $x \wedge y \in \wedge^2 V$ to the endomorphism

$$T_{x\wedge y}(v) = \langle x, v\rangle y - \langle y, v\rangle x \tag{18}$$

is an isomorphism.

Explicitly, matrix commutator corresponds to the operation on $\wedge^2 V$:

$$[u \wedge v, x \wedge y] = \langle u, x\rangle v \wedge y - \langle u, y\rangle v \wedge x - \langle v, x\rangle u \wedge y + \langle v, y\rangle u \wedge x. \tag{19}$$

Thus, with this bracket, we can identify $\wedge^2 V \cong \mathfrak{so}(V)$ as Lie algebras. Notice that we can write

$$[u \wedge v, x \wedge y] = T_{u\wedge v}(x) \wedge y - T_{u\wedge v}(y) \wedge x. \tag{20}$$

Proposition 2.3. *The Lie algebra $\mathfrak{so}(V)$ naturally embeds into the Clifford algebra via the homomorphism*

$$\rho\colon \wedge^2 V \cong \mathfrak{so}(V) \to \mathrm{C}\ell(V, q) \tag{21}$$

defined by

$$\rho(u \wedge v) = \frac{1}{4}(uv - vu). \tag{22}$$

To see that this is a homomorphism we need to see that

$$[\rho(u \wedge v), \rho(x \wedge y)] = \rho\left([u \wedge v, x \wedge y]\right) \tag{23}$$

We first observe the lemma.

Lemma 2.4. *One has $[\rho(u \wedge v), x] = T_{u\wedge v}(x)$ for every $x \in \mathrm{C}\ell(V, q)$.*

PROOF. First, assume that $x \in V$. We use the fundamental identity $uv + vu = -2q(u, v)1$ a few times to see:

$$\begin{aligned}
[\rho(u \wedge v), x] &= \frac{1}{4}\left(uvx - vux - xuv + xvu\right) \\
&= \frac{1}{2}\left(-vux + xvu\right) \\
&= \frac{1}{2}\left(vxu + 2q(u, x)v - vxu - 2q(v, x)u\right) \\
&= q(u, x)v - q(v, x)u \\
&= T_{u\wedge v}(x).
\end{aligned}$$

□

From this lemma we have

$$[\rho(u\wedge v),\rho(x\wedge y)] = T_{u\wedge v}(\rho(x\wedge y)) = \rho(T_{u\wedge v}(x\wedge y)) = \rho\left([u\wedge v, x\wedge y]\right).$$

2.3. Note that by proposition 2.1 that for any $v \in V$ the adjoint action Ad_v preserves the subspace $V \subset C\ell(V,q)$. We define $P(V,q)$ to be the subgroup of $C\ell(V,q)^\times$ generated by vectors $v \in V$ with $q(v) \neq 0$. Let $SP(V,q) = P(V,q) \cap C\ell^{even}(V,q)$. The group $P(V,q), SP(V,q)$ have important subgroups.

Definition 2.5. The ***pin group*** of (V,q) is the subgroup $Pin(V,q) \subset P(V,q)$ generated by elements $v \in V$ with $q(v) = \pm 1$. The ***spin group*** of (V,q) is

$$Spin(V,q) = Pin(V,q) \cap C\ell^{even}(V,q). \tag{24}$$

Explicit presentation for the pin and spin groups are as follows:

$$\begin{aligned} Pin(V,q) &= \{v_1 \cdots v_k \in P(V,q) \mid q(v_j) = \pm 1\ \forall j\} \\ Spin(V,q) &= \{v_1 \cdots v_k \in Pin(V,q) \mid k \text{ even}\} \end{aligned}$$

From proposition 2.1, we recognize that $\mathrm{Ad}_v = -R_v$ where R_v is the reflection across the hyperplane perpendicular to $v \in V$. Define the *twisted* adjoint action

$$\widetilde{\mathrm{Ad}}\colon C\ell(V,q)^\times \to GL\, C\ell(V,q)$$

by the formula

$$\widetilde{\mathrm{Ad}}_\varphi(x) = \alpha(\varphi) x a^{-1}, \tag{25}$$

where α is defined in 1.4. Note that $\widetilde{\mathrm{Ad}}_a$ is *not* an algebra automorphism, but it is still a linear automorphism. Notice that for $v \in V$ one as $\widetilde{\mathrm{Ad}}_v = R_v$ as desired.

Proposition 2.6. *Define*

$$\widetilde{P}(V,q) \overset{\text{def}}{=} \{\varphi \in C\ell(V,q) \mid \operatorname{Im} \widetilde{\mathrm{Ad}}_\varphi = V\}.$$

Then the kernel of the homomorphism

$$\widetilde{\mathrm{Ad}}\colon \widetilde{P}(V,q) \to GL(V)$$

is the group $\mathbf{k}^\times$ *of nonzero multiples of* $1 \in C\ell(V,q)$.

Moreover, $\widetilde{Ad}$ *factors through the group* $O(V,q) \subset GL(V)$.

The next section is dedicated to the proof of this proposition.

2.4. For $a \in C\ell(V,q)$ write $\varphi = \varphi_+ + \varphi_-$ where $\varphi_\pm \in C\ell^{ev/odd}(V,q)$. Then, the condition that $\varphi \in \ker \widetilde{\mathrm{Ad}}$ becomes the pair of equations

$$v\varphi_+ = \varphi_+ v, \quad v\varphi_- = -\varphi_- v. \tag{26}$$

Let $\{e_i\}$ be a basis for V such that $q(e_i) \neq 0$ for all i and $\langle e_i, e_j \rangle = 0$ for all $i \neq j$. Using the fundamental Clifford relation, we see that $\varphi_+ \in C\ell^{ev}(V,q)$ can be expressed in the form $a_0 + e_1 a_1$ where a_0, a_1 are polynomial expressions in the basis elements $e_2, \ldots, e_n$. Since $a_0 + e_1 a_1$ is even we conclude that a_0 is even and a_1 is odd. Applying the relation (26) to $v = e_1$ we see that

$$\begin{aligned} e_1 a_0 + e_1^2 a_1 &= a_0 e_1 + e_1 a_1 e_1 \\ &= e_1 a_0 - e_1^2 a_1. \end{aligned}$$

Thus $e_1^2 a_1 = 0$ and so $a_1 = 0$. This implies that φ_+ is a polynomial expression in $\{e_2, \ldots, e_n\}$. Proceeding iteratively we see that φ is a polynomial expression is *none* of the basis elements, therefore $\varphi_+ \in \mathbf{k} \subset C\ell^{even}(V,q)$. Similarly, one sees that φ_- is an expression in none of the basis elements. But, since φ_- is odd this implies that $\varphi_- = 0$. Since $\varphi \neq 0$ we conclude that $\varphi \in \mathbf{k}^\times$. We have show $\ker \widetilde{\mathrm{Ad}} = \mathbf{k}^\times \subset \widetilde{P}(V,q)$.

To complete the proof we introduce the norm mapping. Let N be the linear endomorphism on the Clifford algebra defined by $N(\varphi) = \varphi \cdot \alpha(\varphi^t)$. Note that

$$\begin{aligned} N(\varphi\psi) &= \varphi\psi\alpha(\psi^t\varphi^t) \\ &= \varphi\psi\alpha(\psi^t)\alpha(\varphi^t) \\ &= \varphi N(\psi)\alpha(\varphi^t). \end{aligned}$$

So, we cannot yet conclude that N is compatible with the algebra structure.

Observe for $v \in V$ that $N(v) = -v^2 = q(v)$. Suppose $\varphi \in \widetilde{P}(V,q)$, so that

$$\alpha(\varphi)v\varphi^{-1} \in V \tag{27}$$

for all $v \in V$. Applying the transpose to this element, which is the identity of course, leads to

$$(\varphi^t)^{-1} v \alpha(\varphi^t) = \alpha(\varphi)v\varphi^{-1}. \tag{28}$$

Rearranging, we see that

$$\begin{aligned} v = \varphi^t \alpha(\varphi) v \varphi^{-1} (\alpha(\varphi^t))^{-1} &= \alpha\left(\alpha(\varphi^t)\varphi\right) v \left(\alpha(\varphi^t)\varphi\right)^{-1} \\ &= \widetilde{\mathrm{Ad}}_{\alpha(\varphi^t)\varphi}(v). \end{aligned}$$

Hence $\alpha(\varphi^t)\varphi \in \ker \widetilde{\mathrm{Ad}} = \mathbf{k}^\times$. We conclude that N factors through the group of units $\mathbf{k}^\times \subset \mathrm{C}\ell(V,q)^\times$:

$$N \colon \widetilde{P}(V,q) \to \mathbf{k}^\times. \tag{29}$$

This finally allows us to see that N is compatible with the algebra structure. Indeed, since $\mathbf{k}^\times$ is in the center of $\mathrm{C}\ell(V,q)$ we have that $N(\varphi\psi) = \varphi N(\psi)\alpha(\varphi^t) = N(\varphi)N(\psi)$.

Notice that $N(\alpha\varphi) = \alpha(\varphi)\varphi^t = N(\varphi)$ for all $\varphi \in \widetilde{P}(V,q)$. Then

$$\begin{aligned} q(\widetilde{\mathrm{Ad}}_\varphi(v)) = N(\widetilde{\mathrm{Ad}}_\varphi(v)) &= N(\alpha(\varphi)v\varphi^{-1}) \\ &= N(\alpha\varphi)N(v)N(\varphi)^{-1} \\ &= q(v). \end{aligned}$$

We conclude that $\widetilde{\mathrm{Ad}}_\varphi$ preserves q for each $\varphi \in \widetilde{P}(V,q)$ so it is an orthogonal transformation.

2.5. By restricting along $P(V,q) \subset \widetilde{P}(V,q)$, proposition 2.6 prescribes a group homomorphism

$$\widetilde{\mathrm{Ad}} \colon P(V,q) \to O(V,q). \tag{30}$$

We study the further restriction to $Pin(V,q)$. The Cartan-Dieudonné theorem implies that the restriction of this homomorphism to $Pin(V,q)$ is surjective. Similarly, the restriction of $\widetilde{\mathrm{Ad}}$ to $Spin(V,q)$ defines a surjective homomorphism

$$\widetilde{\mathrm{Ad}} \colon Spin(V,q) \to SO(V,q). \tag{31}$$

Proposition 2.7. *Suppose* $\mathbf{k} = \mathbf{R}$. *The following sequences are exact*

$$1 \to \mathbf{Z}/2 \to Pin(V,q) \to O(V,q) \to 1 \tag{32}$$

and

$$1 \to \mathbf{Z}/2 \to Spin(V,q) \to SO(V,q) \to 1. \tag{33}$$

PROOF. Cartan and Dieudonné did the hard part of surjectivity. From proposition 2.6 if $a \in P(V,q)$ and $\widetilde{\mathrm{Ad}}_a = \mathbb{1}$ then $a = a_0 1$, $a_0 \in \mathbf{R}^\times$ If a is in $Pin(V,q)$ then we also have $q(a) = \pm 1$, so $a_0 = \pm 1$. The same argument holds for $Spin(V,q)$. □

2.6. Let's focus on the special case $V = \mathbf{R}^n$ with $q = \sum x_i^2$ the standard positive definite inner product. We let $C\ell_n \overset{\text{def}}{=} C\ell(\mathbf{R}^n, \sum x_i^2)$, $SO(n) = SO(\mathbf{R}^n, \sum x_i^2)$, and $Spin(n) = Spin(\mathbf{R}^n, \sum x_i^2)$. By the above, for $n \geq 3$ there is a short exact sequence of Lie groups

$$1 \to \mathbf{Z}/2 \to Spin(n) \to SO(n) \to 1. \tag{34}$$

THEOREM 2.8. *For $n \geq 3$, the exact sequence (34) represents the universal double cover of $SO(n)$.*

Recall that the universal double cover of a connected topological group G is a covering space

$$1 \to \pi_1(H) \to G \to H \to 1 \tag{35}$$

where G is the group of equivalence classes of homotopy classes of paths in H with pointwise multiplication. For Lie groups, the universal cover is even more constrained. A basic fact from Lie theory is that any Lie algebra $\mathfrak{g}$ is the Lie algebra of a simply connected Lie group G. Thus, the universal cover of a connected Lie group H is a simply connected Lie group G together with a homomorphism $\rho\colon G \to H$ which induces an isomorphism at the level of Lie algebras. Already from the short exact sequence of Lie groups in proposition 2.7 we see that $\widetilde{\mathrm{Ad}}$ induces an isomorphism at the level of Lie algebras.

To prove the theorem we proceed with the following steps.

(1) First, we will show that $\pi_1(SO(n)) = \pi_1(SO(n+1))$ for $n \geq 3$.
(2) Next, we will show that $\pi_1(SO(3)) = \mathbf{Z}/2$ by showing $Spin(3) = SU(2)$.
(3) Finally, we will argue that $\pi_1(Spin(n)) = 0$ for $n \geq 3$, thus completing the proof.

For (1) we use the long exact sequence in homotopy groups associated to the fibration

$$SO(n) \hookrightarrow SO(n+1) \to S^n \tag{36}$$

induced from embedding $SO(n)$ block diagonally into $SO(n+1)$.

For (2) we first construct an explicit isomorphism between the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ and $\mathbf{R}^3$. Explicitly $\mathfrak{su}(2)$ is the Lie algebra of complex 2×2 matrices of the form

$$\begin{pmatrix} a & -\bar{b} \\ b & -a \end{pmatrix}. \tag{37}$$

In terms of matrices, the standard positive definite inner product on $\mathbf{R}^3$ is

$$\frac{1}{2}\operatorname{Tr}(XY^\dagger). \tag{38}$$

Now, define the homomorphism

$$\rho\colon SU(2) \to GL(\mathfrak{su}(2)) \cong GL(\mathbf{R}^3) \tag{39}$$

by sending the matrix A to $X \mapsto AXA^{-1}$. This matrix factors through $SO(3)$. It is surjective and its kernel is $\mathbf{Z}/2$. Since $SU(2)$ is diffeomorphic to S^3 it follows that $Spin(3) \cong SU(2)$ and $\pi_1(SO(3)) =$ With this identification $\rho = \widetilde{\mathrm{Ad}}$.

Finally, to see (3) we recall that covering space theory implies that $\pi_1(Spin(n))$ is a finite index sugroup of $\mathbf{Z}/2$. So, for $n \geq 3$ one sees that $\pi_1(Spin(n))$ is trivial.

3. Low-dimensional examples

We will present some basic low-dimensional examples of real Clifford algebras and spin groups. We let $C\ell_{r,s}$ denote the Clifford algebra of the vector space $\mathbf{R}^{r+s}$ associated to the quadratic form of signature (r,s). (In terms of our previous notation $C\ell_{n,0} = C\ell_n$.)

3.1. The Clifford algebra $C\ell_1$ is generated by elements $1, e$ with the relation $e^2 = -1$. Thus $C\ell_1 \cong \mathbf{C}$ as real associative algebras. Under this identification, $C\ell_1^{ev} = \mathbf{R}$ and $C\ell_1^{odd} = \mathrm{i}\mathbf{R}$. The transpose operation is the identity. The map α is complex conjugation $\alpha(z) = \bar{z}$. The group of units is the nonzero complex numbers under multiplication $C\ell_1^\times = \mathbf{C}^\times$. The norm map is $N(z) = z\bar{z}$.

We know from the exact sequences from proposition 2.7 that

$$Pin(1) \simeq \mathbf{Z}/4, \quad Spin(1) \simeq \mathbf{Z}/2. \tag{40}$$

Let's see this explicitly. Per the isomorphisms of the previous section, we can identify $Pin(1)$ with the group of elements $z = a + \mathrm{i}b \in \mathbf{C}^\times$ such that $a = \pm 1, b = 0$ or $a = 0, b = \pm 1$. Thus $Pin(1) = \{1, -1, \mathrm{i}, -\mathrm{i}\} = \mathbf{Z}/4$ and $Spin(1) = \{1, -1\} = \mathbf{Z}/2$.

3.2. Next we look at $C\ell_2$. Let $\{e_1, e_2\}$ be an orthonormal basis for $V = \mathbf{R}^2$. Then $C\ell_2$ is spanned by the basis $\{1, e_1, e_2, e_1e_2\}$ subject to the relations

$$e_1e_2 = -e_1e_2, \quad e_1^2 = e_2^2 = -1, \quad (e_1e_2)^2 = -1. \tag{41}$$

Define the real linear map

$$\Phi\colon C\ell_2 \to \mathbf{H} \tag{42}$$

by the rules $e_1 \mapsto \mathsf{i}, e_2 \mapsto \mathsf{j}, e_1e_2 \mapsto \mathsf{k}$. It is immediate to check that this is an isomorphism of real algebras. Thus $C\ell_2$ is isomorphic to the quaternions, which is of course generated over $\mathbf{R}$ by $\{1, \mathsf{i}, \mathsf{j}, \mathsf{k}\}$ satisfying the usual conditions.

In quaternion terms the transpose is

$$1^t = 1, \quad \mathsf{i}^t = \mathsf{i}, \quad \mathsf{j}^t = \mathsf{j}, \quad \mathsf{k}^t = -\mathsf{k}. \tag{43}$$

The involution α is

$$\alpha(1) = 1, \quad \alpha(\mathsf{i}) = -\mathsf{i}, \quad \alpha(\mathsf{j}) = -\mathsf{j}, \quad \alpha(\mathsf{k}) = \mathsf{k}. \tag{44}$$

In particular, $1, \mathsf{k}$ are even and i, j are odd. The norm is

$$N(1) = N(\mathsf{i}) = N(\mathsf{j}) = N(\mathsf{k}) = 1. \tag{45}$$

The group $Pin(2)$ thus consists of elements

$$a1 + b\mathsf{i} + c\mathsf{j} + d\mathsf{k}, \quad a, b, c, d \in \mathbf{R} \tag{46}$$

such that

- Either $b = c = 0$ and $a^2 + d^2 = 1$, or
- $a = d = 0$ and $b^2 + c^2 = 1$.

We conclude that $Pin(2) \simeq U(1) \sqcup U(1)$ and $Spin(2) \simeq U(1)$.

In quaternion notation, the group $Spin(2) \simeq U(1)$ consists of elements $a1 + d\mathsf{k} \subset \mathbf{H}$ satisfying $a^2 + d^2 = 1$. In terms of a real orthonormal basis of $\mathbf{R}^2$, this group is presented as the elements

$$x = a1 + be_1e_2 \tag{47}$$

satisfying $N(x) = a^2 + b^2 = 1$.

3.3. The Clifford algebra $Cl_{0,2}$ is spanned by vectors $1, x, y, xy$ which satisfy

$$x^2 = y^2 = 1, \quad xy = -yx. \tag{48}$$

The correspondence

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$y \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

sets up an isomorphism of algebras betwween $Cl_{0,2}$ and the algebra of 2×2 real matrices.

For future reference, if $\mathbf{k}$ is a field, we will denote $\mathbf{k}(n)$ for the algebra of $n \times n$ matrices with coefficients in $\mathbf{k}$.

Thus $\mathrm{C}\ell_{0,2} \cong \mathbf{R}(2)$.

3.4. The Clifford algebra $\mathrm{C}\ell_{1,1}$ is spanned by vectors $1, x, y, xy$ which satisfy

(49) $$x^2 = -y^2 = 1, \quad xy = -yx.$$

The correspondence

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$y \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

sets up an isomorphism of algebras betwween $Cl_{1,1}$ and the algebra of 2×2 real matrices.

4. Classification of Clifford algebras

The main result of this section is to prove the following periodicity result for Clifford algebras

(50) $$\mathrm{C}\ell_{n+4} \simeq \mathrm{C}\ell_n \otimes \mathrm{C}\ell_4.$$

4.1.

Proposition 4.1. *There are isomorphism*

$$
\begin{aligned}
C\ell_{n+2,0} &\simeq C\ell_{0,n} \otimes C\ell_{2,0} \\
C\ell_{0,n+2} &\simeq C\ell_{n,0} \otimes C\ell_{0,2} \\
C\ell_{n+1,m+1} &\simeq C\ell_{n,m} \otimes C\ell_{1,1}.
\end{aligned} \tag{51}
$$

The tensor product is the ordinary (ungraded) tensor product of algebras.

PROOF. Let $\{e_i\}$ denote an orthonormal basis for $\mathbf{R}^{n+2}$ with respect to the standard positive definite form. Let $\{e'_i\}$ be an orthornormal basis for $\mathbf{R}^n$ which we view as generators for the algebra $C\ell_{0,n}$. In particular $e'_i e'_i = 1$ (note the lack of sign). Let $\{e''_1, e''_2\}$ be an orthonormal basis for $\mathbf{R}^2$ which we view as generators for $C\ell_{2,0}$. Define the linear map

$$
f\colon \mathbf{R}^{n+2} \to C\ell_{0,n} \otimes C\ell_{2,0} \tag{52}
$$

by the following rules. If $i = 1, \ldots, n$ then define $f(e_i) = e'_i \otimes e''_1 e''_2$. Additionally $f(e_{n+1}) = 1 \otimes e''_1$ and $f(e_{n+2}) = 1 \otimes e''_2$. This map f satisfies $f(v)^2 = -\|v\|^2$ where $\| - \|$ is the ordinary norm on $\mathbf{R}^{n+2}$. Thus f extends to a homomorphism $\widetilde{f}\colon C\ell_{n+2,0} \to C\ell_{0,n} \otimes C\ell_{2,0}$. Since f hits all generators it follows that $\widetilde{f}$ is surjective. By counting dimensions we see its an isomorphism. $\square$

4.2. Using the proposition of the previous section we can prove the stated periodicity result.

THEOREM 4.2. *There is an isomorphism of real algebras*

$$
C\ell_{n+4} \simeq C\ell_n \otimes C\ell_4 \tag{53}
$$

for every n. The tensor product is the ordinary (ungraded) tensor product.

We note that there are similar periodicity results for the non-definite signature Clifford algebras.

PROOF. From the proposition of the previous subsection

$$
C\ell_{4,0} \simeq C\ell_{0,2} \otimes C\ell_{2,0} \simeq \mathbf{H} \otimes \mathbf{R}(2) \simeq \mathbf{H}(2) \simeq C\ell_{2,0} \otimes C\ell_{0,2} \simeq C\ell_{0,4} \tag{54}
$$

Thus using the proposition again we have

$$
\begin{aligned}
C\ell_n \otimes C\ell_{4,0} &\simeq C\ell_n \otimes C\ell_{0,2} \otimes C\ell_{2,0} \\
&\simeq C\ell_{0,n+2} \otimes C\ell_{2,0} \\
&\simeq C\ell_{n+4,0}
\end{aligned} \tag{55}
$$

as desired. ☐

4.3. From the theorem we see that to describe explicitly $C\ell_n$ for any n it suffices to know $C\ell_0 = \mathbf{k}, C\ell_1 = \mathbf{C}, C\ell_2 = \mathbf{H}, C\ell_3$ and $C\ell_4$. We already computed $C\ell_4 = \mathbf{H}(2)$ in the proof of the theorem. Finally $C\ell_3 = C\ell_{0,1} \otimes C\ell_{2,0} = \mathbf{H} \oplus \mathbf{H}$.

References

[LM89] H. B. Lawson Jr. and M.-L. Michelsohn. *Spin geometry*. Vol. 38. Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1989, pp. xii+427.