February 1

Suppose $\{e_i\}$ is an orthonormal frame for T_M, then

$$\Delta^E = -\nabla^E_{e_i}\nabla^E_{e_i} + \nabla^E_{\nabla_{e_i} e_i}.$$

Or, if $\{x^i\}$ is any smooth coordinate, then

$$\Delta^E = - g^{ij}(x)\left(\nabla^E_{\partial_i}\nabla^E_{\partial_j} - \Gamma^k_{ij}\nabla^E_{\partial_k}\right)$$

where $\{\Gamma^k_{ij}\}$ are christoffel symbols

$$\nabla_{\partial_i}\partial_k = \Gamma^k_{ij}\partial_k.$$

- The standard or "scalar" Laplacian is the case E is trivial and $\nabla^E = d$. Then

$$\Delta f = -\operatorname{Tr}(\nabla df).$$

or in local coordinates:

$$\Delta f = - g^{ij}(x) \left(\partial_i \partial_j - \Gamma^k_{ij} \partial_k \right) f .$$

- We will show that any generalized Laplacian H is of the form

$$H = \Delta^E + F$$

where $F \in \Gamma(M, \operatorname{End} E)$

<u>Prop</u>: If H is generalized Laplacian, there exists a connection ∇^E s.t. $\forall f \in C^\infty(M)$:

$$[H, f] = -2 \langle \operatorname{grad} f, \nabla^E \rangle + \Delta f ,$$

as first-order differential operators acting on $\Gamma(M, E)$.

Pf: Define ∇^E by the formula

$$\langle f_0 \,\mathrm{grad}\, f_1 , \nabla^E s \rangle \overset{\mathrm{dfn}}{=} \frac{1}{2} f_0 \left(-H(f_1 s) + f_1 Hs + (\Delta f_1) s \right)$$
$$= \frac{1}{2} f_0 \left(-[H, f_1] s + (\Delta f_1) s \right).$$

Claim: ∇^E is well-defined.

[Sps $f_0, f_1, f_2 \in C^\infty(M)$. Then

$$f_0 \,\mathrm{grad}\, f_1 f_2 = f_0 f_1 \,\mathrm{grad}\, f_2 + f_0 f_2 \,\mathrm{grad}\, f_1 .$$

Now: $[[H, f_1], f_2] s = -2 (df_1, df_2) s$

$$\begin{gathered} [[H, f_1], f_2] s \\ \| \\ [H f_1 - f_1 H , f_2] s \\ \| \\ H(f_1 f_2 s) - f_1 H(f_2 s) - f_2 H(f_1 s) \\ + f_1 f_2 H s . \end{gathered}$$

$\Rightarrow$

$$\langle f_0 \operatorname{grad} f_1 f_2 , \nabla^E s \rangle$$

$$= \langle f_0 f_2 \operatorname{grad} f_1 + f_0 f_1 \operatorname{grad} f_2 , \nabla^E s \rangle \quad]$$

<u>Claim</u>: $\nabla^E (h s) = dh \otimes s + h \nabla^E s$.

$$([\langle \operatorname{grad} f , \nabla^E \rangle , h]$$

$$= [-\tfrac{1}{2} [H , f] + \tfrac{1}{2} \Delta f , h]$$

$$= -\tfrac{1}{2} [[H , f] , h] = \langle \operatorname{grad} f , dh \rangle . \quad \square$$

Note: the operator

$$F \overset{dfn}{=} H - \Delta^E$$

satisfies

$$[F , f] = [H , f] - [\Delta^E , f]$$

$$= -2\langle \operatorname{grad} f, \nabla^E \rangle - (-2\langle \operatorname{grad} f, \nabla^E \rangle).$$

$$= 0.$$

$\Rightarrow$ F is a zeroth order operator.

<u>Summary</u>: A generalized Laplacian is determined by:

① A metric g on M, this determines the second-order piece.

② A connection ∇^E on E, this determines the first-order piece.

③ A section F of $\operatorname{End} E$, this is the zeroth order piece.

<u>Densities</u>: M any smooth manifold. Let

$$\mathrm{Dens}_M \overset{\mathrm{dfn}}{=} \mathrm{Fr}_M \overset{GL_n}{\times} \mathbb{R}_{|\det|^{-1}}$$

where $\mathbb{C}_{|\det|}$ is the 1-dim'l rep^n

$$GL_n \longrightarrow \mathbb{R}^\times$$
$$A \longmapsto |\det A|^{-1}.$$

The line bundle Dens_M is always trivializable. Nowhere vanishing section $|dx|$:

$$|dx|(\partial_1 \wedge \cdots \wedge \partial_n) = 1.$$

gluing is a choice.

Integration:

$$\int_M : \Gamma_c(M, \mathrm{Dens}_M) \to \mathbb{R}.$$

In local coordinates

$$\int_M f(x)\,|dx| = \int_{\mathbb{R}^n} f(x)\,dx^1 \wedge \cdots \wedge dx^n .$$

More generally, for $s \in \mathbb{R}$ let

$$\mathrm{Dens}^s_M = \mathrm{Fr}_M \times^{GL_n} \mathbb{R}_{|\det|^{-s}} .$$

An <u>orientation</u> is a choice of isomorphism

$$\mathrm{Dens}_M \overset{1-1}{\cong} \Lambda^n T^*_M .$$

In this case define $\displaystyle\int_M \alpha = \int_M |\alpha|$

- If E is any vector bundle, then is a pairing

$$\Gamma_c(M, E) \times \Gamma(M, E^* \otimes \mathrm{Dens}_M) \longrightarrow \mathbb{R}$$

$$(\alpha, \beta) \longmapsto \int_M \langle \alpha, \beta \rangle$$

- A Hermitian vector bundle is a complex vector bundle E w/ a fiberwise Hermitian inner product that varies smoothly.

In this case $\Gamma_c(M, E \otimes \mathrm{Dens}_M^{1/2})$ has a natural inner product.

Dfn : 1) Let E_1, E_2 be v.b.'s and

let $D : \Gamma(M, E_1) \to \Gamma(M, E_2)$ be a differential operator. The formal adjoint

$$D^* : \Gamma(M, E_2^* \otimes \mathrm{Dens}_M) \to \Gamma(M, E_1^* \otimes \mathrm{Dens}_M)$$

is :

$$\int_M \langle D\alpha, \beta \rangle = \int_M \langle \alpha, D^*\beta \rangle ,$$

where $\alpha \in \Gamma_c(M, E_1)$, $\beta \in \Gamma(M, E_2^* \otimes \mathrm{Dens}_M)$.

2) If E is Hermitian and

$$D \in \mathcal{D}(M, E \otimes \mathrm{Dens}^{1/2})$$

we say D is symmetric if

$$D = D^* .$$