February 13

We begin our dive into the construction of Atiyah-Bott-Shapiro.

$$\begin{tikzcd} \mathcal{M}_n = \text{Grothendieck group of } \mathbb{Z}/2 \text{ graded Clifford modules.} \arrow[d, "\text{forget}"'] \\ \overline{\mathcal{M}}_n = \text{Grothendieck group of (ungraded) Clifford modules.} \end{tikzcd}$$

We have seen that

$$\mathcal{M}_n = \begin{cases} \mathbb{Z} \,, & n \neq 0,4 \bmod 8 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 0,4 \bmod 8 . \end{cases}$$

$$\overline{\mathcal{M}}_n = \begin{cases} \mathbb{Z} \,, & n \neq 3,7 \bmod 8 \\ \mathbb{Z} \oplus \mathbb{Z} \,, & n = 3,7 \bmod 8 . \end{cases}$$

Prop: There is an equivalence of categories

$$\begin{cases} R : \mathrm{Mod}^{gr}_{Cl_n} \xrightarrow{\sim} \mathrm{Mod}_{Cl_n^0} . \\ M = \underset{=}{M^0} \oplus \underset{=}{M^1} \longmapsto M^0 . \end{cases}$$

Pf: We define $S : \mathrm{Mod}_{Cl_n^0} \to \mathrm{Mod}^{gr}_{Cl_n}$

by $S(N) = {}_{Cl_n}\circlearrowright\, Cl_n \otimes_{Cl_n^0} N$

the grading is induced from the one on Cl_n.

Then

$$\begin{tikzcd} S \circ R(M) = S(M^0) \arrow[d, "\eta"] & = Cl_n \otimes_{Cl_n^0} M^0 \arrow[dl, "\cong"'] \\ M & \end{tikzcd}$$

$$\underset{=}{x} \otimes \underset{=}{m} \longmapsto x \cdot m .$$

$$R \circ S(N) = R(Cl_n \otimes_{Cl^0} N) \cong N .$$

Hint:

$\{e_i\}$ o.n.b. $\quad e_i \cdot (-) : M^0 \xrightarrow{\sim} M^1$

∎

Thus

$$\mathcal{M}_n \xrightarrow[\cong]{R} \left\{ \begin{array}{c} \text{Grothendieck group of} \\ Cl_n^0\text{-modules.} \end{array} \right\}$$

The coordinate inclusion $\mathbb{R}^n \overset{i}{\hookrightarrow} \mathbb{R}^{n+1}$ induces

$$\mathcal{M}_{n+1} \xrightarrow{i^* = \text{restriction.}} \mathcal{M}_n .$$

<u>Define</u>: $A_n = \operatorname{coker}\left(\mathcal{M}_{n+1} \xrightarrow{i^*} \mathcal{M}_n\right).$

<u>Thm</u>:

$$\begin{bmatrix} n \neq 4k \\ n = 4k \end{bmatrix}.$$

$$A_n \cong \begin{cases} \mathbb{Z}/2, & n = 1 \\ \mathbb{Z}/2, & n = 2 \\ 0, & n = 3 \\ \mathbb{Z}, & n = 4 \\ 0, & n = 5 \\ 0, & n = 6 \\ 0, & n = 7 \\ \mathbb{Z}, & n = 8 \end{cases} \quad \text{mod } 8 .$$

$$\begin{tikzcd} \bullet\ \underline{n=1}: & Cl_2 \cong \mathbb{H} \cong \langle 1, e_1, e_2, e_1e_2 \rangle / \text{usual} \\ & Cl_1 \cong \mathbb{C} \cong \langle 1, e_1 \rangle / \text{usual.} \arrow[u, "i"] \end{tikzcd}$$

$e_1^2 = -1,\ e_2^2 = -1,\ e_1e_2 = -e_2e_1$

$i(1) = 1,\ i(e_1) = e_1$ $\qquad$ $e_1^2 = -1.$

$i^* \mathbb{H}$ is the Cl_1-module generated by $1, e_1, e_2, e_1e_2$

$$\cong \mathbb{C}\{1, e_1\} \oplus \mathbb{C}\{e_2, e_1e_2\}$$

$$\mathbb{C} \oplus \mathbb{C} \cong \qquad \underset{e_1}{\circlearrowleft} \qquad\qquad \underset{e_1}{\circlearrowleft}$$

$$\Rightarrow \boxed{A_1 \cong \mathbb{Z}/2.}$$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, dashed, "2"] \arrow[d, equal] & \mathbb{Z} \arrow[d, equal] \\ A_1 = \operatorname{coker}\big(\mathcal{M}_2 \arrow[r, "i^*"'] & \mathcal{M}_1\big). \end{tikzcd}$$

$$x \longmapsto 2y$$

$$\mathcal{M}_2 = \mathbb{Z} \cdot x, \quad \mathcal{M}_1 = \mathbb{Z} \cdot y$$

Similar computation of A_n when $n \neq 4k$.

Now, we compute A_{4k}. Recall that

$$\mathcal{M}_{4k} = \mathbb{Z} \oplus \mathbb{Z}.$$

If $M = M^0 \oplus M^1$ define M^* to be the gr. module $(M^*)^{0,1} = M^{1,0}$.

[If $n \neq 4k$ then $M \cong M^*$.]

Prop: If M, N are the two inequivalent gr. irreps of Cl_{4k}. Then:

$$M^* = N, \quad N^* = M.$$

Pf: Let $\phi : Cl_{n-1} \xrightarrow{\cong} Cl_n^0$, $e_i \mapsto e_i e_n$.

$\gamma : Cl_n \longrightarrow Cl_n$, $x \mapsto e_n x e_n^{-1}$

$\alpha : Cl_{n-1} \longrightarrow Cl_{n-1}$, $x \mapsto (-1)^{|x|} x$.

Then the following diagram commutes

$$\begin{tikzcd}
Cl_{n-1} \arrow[r, "\phi"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowright"] & Cl_n^0 \arrow[d, "\gamma"] \\
Cl_{n-1} \arrow[r, "\phi"'] & Cl_n^0
\end{tikzcd}$$

<u>Claim</u>: For $M \in Mod_{Cl_n}$ let M^α be the new module $x \cdot m = \alpha(x) \cdot m$. Then

$$M^* \cong M^\alpha .$$

[Note $e_n \cdot (-) : M^0 \xrightarrow{\cong} M^1$ iso of plain vector spaces.]

It follows:

$$\begin{tikzcd}
\mathcal{M}_n \arrow[d, phantom, "\cong"] & M \arrow[r, mapsto] \arrow[d, mapsto] & M^* \arrow[d, mapsto] \\
\mathcal{M}(Cl_n^0) & M^0 \arrow[r, mapsto] & (M^0)^\alpha
\end{tikzcd}$$

So to complete the proof we need to show β swaps the two ungraded reps of

$$\boxed{Cl_{n-1} = Cl_{4k-1}}.$$

▨

<u>Cor</u>: $A_{4k} \cong \mathbb{Z}$. ~ Two irreps.

<u>Pf</u>: Let $x, y \in M_{4k}$ be generators.

$$z \in M_{4k+1}.$$

Since $z^* = z \Rightarrow$

$$(i^* z)^* = i^* z^* = i^* z \Rightarrow \boxed{i^* z = x + y.}$$

▨

$$\begin{tikzcd}
M_{4k+1} \arrow[r, "i^*"] \arrow[d, equal] & M_{4k} \arrow[d, equal] & \\
\mathbb{Z} \arrow[r, "\mathrm{diag}"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z}.
\end{tikzcd}$$

$A_{4k} \cong \mathbb{Z}$

- <u>Ring structure</u>: Let

$$\boxed{\mathcal{M}_* = \bigoplus_{n \geq 0} \mathcal{M}_n}, \quad \boxed{A_* = \bigoplus_{n \geq 0} A_n.}$$

Recall that there is an isomorphism

$$\phi_{k,l} : Cl_{k+l} \xrightarrow{\cong} Cl_k \hat{\otimes}_{\mathbb{R}} Cl_l .$$

$$\phi_{k,l}(e_i) = \begin{cases} e_i \otimes 1, & 1 \leq i \leq k, \\ 1 \otimes e_i, & k < i \leq k+l. \end{cases}$$

So if M is gr. Cl_k-mod, N is gr. Cl_l-mod then

$$\phi_{k,l}^*\left(M \overset{gr}{\otimes} N\right) \in Cl_{k+l}\text{-mod}.$$

<u>Prop</u>: This defines an associative multiplication

$$\mathcal{M}_k \times \mathcal{M}_l \longrightarrow \mathcal{M}_{k+l} .$$

• Sps $u \in \mathcal{M}_k$, $v \in \mathcal{M}_\ell$. Then

$u = [M]$, $v = [N]$, for some gr. modules M, N.

Have $(u \cdot v)^* = [\phi^*_{k,\ell}(M \overset{gr}{\otimes} N)]^*$

$$= \left[\phi^*_{k,\ell}\left(\underbrace{M^0 \otimes N^0 \oplus M^1 \otimes N^1}_{\text{even}} \oplus \underbrace{M^0 \otimes N^1 \oplus M^1 \otimes N^1}_{\text{odd}}\right)\right]^*$$

$$= \left[\phi^*_{k,\ell}\left(\underbrace{M^0 \otimes N^1 \oplus M^1 \otimes N^0}_{\text{even}} \oplus \underbrace{M^0 \otimes N^0 \oplus M^1 \otimes N^1}_{\text{odd}}\right)\right]$$

$$= \left[\phi^*_{k,\ell}\left((\underset{\text{ev}}{M^0} \oplus \underset{\text{odd}}{M^1}) \overset{gr}{\otimes} (\underset{\text{ev}}{N^1} \oplus \underset{\text{odd}}{N^0}]\right)\right]$$

$= u v^*$.

$\Rightarrow \quad (uv)^* = uv^*$.

(Similarly $(uv)^* = u^* v$.)

Sps $\lambda \in \mathcal{M}_8 \cong \mathbb{Z}$ is a generator.

Prop: $\lambda \cdot (-) : \mathcal{M}_k \xrightarrow{\cong} \mathcal{M}_{k+8}$.

Pf: $k \neq 4l$. Then let $x \in \mathcal{M}_k$ be the class of unique irrep.

$$\dim (\lambda \cdot x)^0 = \dim \lambda^0 \cdot \dim x^0 + \dim \lambda^1 \cdot \dim x^1 .$$

$$= 2 \dim \lambda^0 \dim x^0 .$$

We have $\dim \lambda^0 = 8$ and hence

$$\dim (\lambda \cdot x)^0 = 16 \dim x^0 = \dim \begin{pmatrix} \text{irrep of} \\ Cl_{k+8} \end{pmatrix}^0 .$$

Now, suppose that $k = 4l$. There are two generators of $\mathcal{M}_{4l}$ call x, y. We know $y^\alpha = x$. As above we see that $\lambda \cdot x$ is one generator of $\mathcal{M}_{4l+8}$. But $\lambda \cdot y = \lambda \cdot x^\alpha = (\lambda \cdot x)^\alpha$. so this is the other one. $\square$

· Let $i : Cl_{n-1} \hookrightarrow Cl_n$. Then

$$i^*(uv) = u\, i^* v.$$

Thus, we see that im $i^* \subseteq \mathcal{M}_\bullet$ is an ideal. Thus we also get a ring str. on $A_\bullet$.

$$A_k \times A_l \longrightarrow A_{k+l}.$$

$\bigoplus_{n\geq 0} A_n$

$\|$

Thm: $A_\bullet \cong \dfrac{\mathbb{Z}[\overset{1}{\xi}, \overset{4}{\mu}, \overset{8}{\lambda}]}{(2\xi, \xi^3, \xi\mu, \mu^2 - 4\lambda)}$. (deg 1 4 8)

On HW you will show:

Thm: $A^{\mathbb{C}}_\bullet \cong \mathbb{Z}[\alpha] \leftarrow$ describe this geometrically

$\uparrow$ deg 2.

$\|$

$\bigoplus_{n\geq 0} A^{\mathbb{C}}_n$