

February 15 |

We will construct / define the heat kernel
of a generalized Laplacian on

$$E \otimes \text{Dens}^{1/2}$$

our compact, Riemannian M .

• Schwartz kernels

Def: The space of distributional (or
generalized) sections of E is:

$$\overline{\Gamma}(M, E) \stackrel{\text{def}}{=} \left(\Gamma(M, E \otimes \text{Dens}) \right)^{\vee}$$

\uparrow
top^l dual.

Remark: Have embedding

$$\Gamma(M, E) \hookrightarrow \overline{\Gamma}(M, E).$$

Ex: Sp's $E = \text{Dens}$ Have 'Dirac' distributions

$$\delta_p \in \bar{\Gamma}(M, \text{Dens})$$

defined by:

$$\delta_p : f \in C^\infty(M) \mapsto f(p)$$

• If \bar{E}_1, \bar{E}_2 v.s.'s then let

$$\begin{array}{ccc} & \searrow \downarrow & \\ & H & E_1 \otimes E_2 \\ & \uparrow \uparrow & \parallel \cong \\ p_1 \quad p_2 & M \times M & p_1^* E_1 \otimes p_2^* E_2 \end{array}$$

A kernel is a section

$$K \in \Gamma(M \times M, (F \otimes \text{Dens}^{1/2}) \otimes (E^* \otimes \text{Dens}^{1/2}))$$

Such a kernel defines an operator

$$K : \bar{\Gamma}(M, \bar{E} \otimes \text{Dens}^{1/2}) \rightarrow \Gamma(M, F \otimes \text{Dens}^{1/2})$$

by

$$(Ks)(x) = \int_{y \in M} k(x, y) s(y)$$

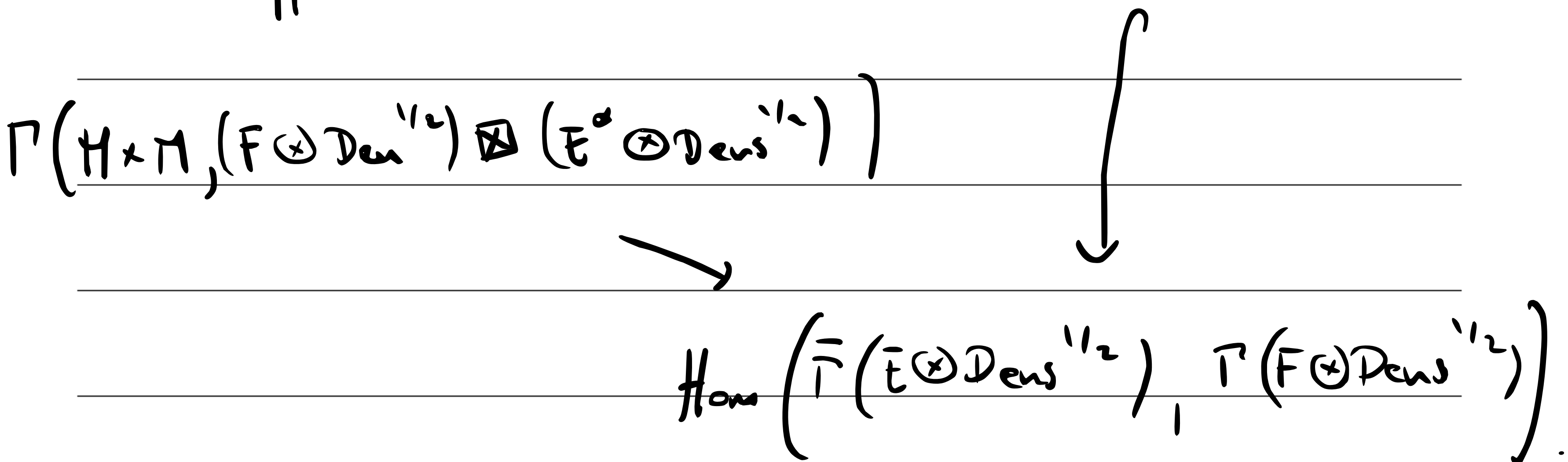
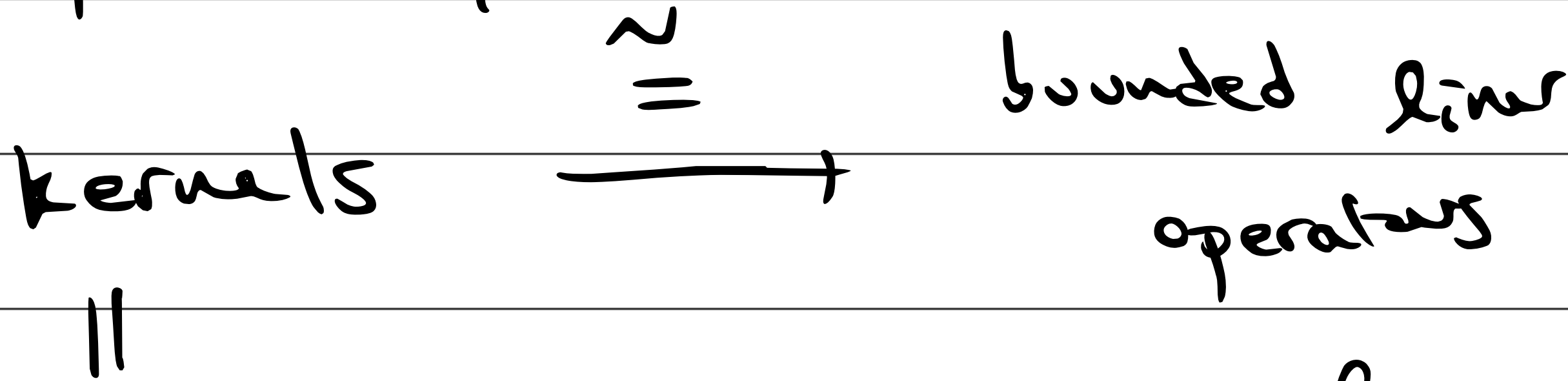
If k_1, k_2 are two kernels s.t.

K_1, K_2 are composable

then $K_1 \circ K_2$ is the operator w/ kernel

$$k(x, y) = \int_{z \in M} k_1(x, z) k_2(z, y)$$

Thm: [Schwarz]



• Dirac notation: If \mathbb{P} is any bounded linear operator then we denote

$$p(x, y) = \langle x | \mathbb{P} | y \rangle$$

its associated kernel. Thus

$$(\mathbb{P}s)(x) = \int_{y \in H} \langle x | \mathbb{P} | y \rangle s(y).$$

(family of)

We are interested in the operator:

$$e^{-tH}, \quad t > 0$$

where $H =$ generalized Laplacian.

Dfn: let H be generalised Laplacian on $E \otimes \text{Dens}^{1/2}$

A heat kernel for H is

$$P_t(x, y) \in \Gamma_{\text{cts}}(\mathbb{R}_+ \times M \times M, (E \otimes \text{Dens}^{1/2}) \otimes (E \otimes \text{Dens}^{1/2}))$$

st.

- ① $P_t(x, y)$ is C^1 wrt t .
- ② $P_t(x, y)$ is C^2 wrt \underline{x} , for any coordinate \underline{x} of M .
- ③ $(\partial_t + H_x) P_t(x, y) = 0$.
- ④ $\lim_{t \rightarrow 0} P_t S = S$.

in other words

$$\lim_{t \rightarrow 0} P_t(x, y) = \delta(x - y).$$

We show such a heat kernel is unique.

Lemma: Assume the formal adjoint H^* has a heat kernel, p_t^a . If

$$s_t: \mathbb{R}_+ \rightarrow \Gamma(E \otimes \text{Dens}^{1/2})$$

is C^1 in t , C^2 in x

$$\lim_{t \rightarrow 0} s_t = 0, \quad (\partial_t + H)s_t = 0$$

$$\Rightarrow s_t = 0.$$

Pf: If $s_1(t, -), s_2(t, -)$ are C^2 sections

of $E \otimes \text{Dens}^{1/2}$ then

$$\int_M \langle Hs_1(t, -), s_2(t, -) \rangle$$

$$= \int_M \langle s_1(t, -), Hs_2(t, -) \rangle.$$

$$\forall t > 0.$$

For $u \in \Gamma(E^{\otimes} \otimes \text{Dens}^{\wedge 1/2})$ define

$$f_u(\theta) = \int_{(x,y) \in M \times M} \langle s(\theta, x), p_{t-\theta}^{\otimes}(x,y) u(y) \rangle.$$

$$f_u \quad 0 < \theta < t.$$

$$\frac{\partial}{\partial \theta} f_u(\theta) = \int_{(x,y)} \langle \partial_{\theta} s(\theta, x), p_{t-\theta}^{\otimes}(x,y) u(y) \rangle$$

$$+ \int_{(x,y)} \langle s(\theta, x), H^{\otimes} p_{t-\theta}^{\otimes}(x,y) u(y) \rangle.$$

$$= \int_{(x,y)} \langle (\partial_{\theta} + H) s(\theta, x), p_{t-\theta}^{\otimes}(x,y) u(y) \rangle$$

$$= 0 \quad \text{by Heat eqn.}$$

$\Rightarrow f_u(\theta)$ is constant.

$$\lim_{\theta \rightarrow t} f_u(\theta) = \int_H \langle s(\theta, x), u(x) \rangle$$

$$\text{But } \lim_{\theta \rightarrow 0} f_u(\theta) = 0. \Rightarrow$$

$$\int_H s(t, x) u(x) = 0 \quad \forall t > 0.$$

$$\forall u \in \Gamma(E^* \otimes \text{Dens}^{1/2}). \Rightarrow s(t, -) = 0. \quad \square$$

Cor: 1) Sps $\exists p_t^a$ for H^a , then $\exists p_t$ for H .

2) Sps $\exists p_t^a, p_t$ for H^a, H . Then

$$p_t(x, y) = (p_t(y, x))^a.$$

3) Sps $\exists p_t^a, p_t$ for H^a, H . Then the

operators $P_t s = \int_H p_t s$ form a semi-group.

Pf: Let

$$f_u(\theta) = \int_H ((P_\theta s)(z), (P_{t-\theta}^* u)(z))$$

As before, f is constant \Rightarrow

$$(P_t s, u) = (s, P_t^* u)$$

This proves 1, 2).

Semigroup $P_{t+s} = P_t P_s$.

$$\lim_{t \rightarrow 0} s_t = P_\theta s \Rightarrow s_t = P_{t+\theta} s$$

by lemma.

