February 15

We will construct / define the heat kernel of a generalized Laplacian on

$$E \otimes \mathrm{Dens}^{1/2}$$

over compact, Riemannian M.

- Schwartz kernels

Def: The space of distributional (or generalized) sections of E is:

$$\overline{\Gamma}(M, E) \overset{\text{dfn}}{=} \underbrace{\left(\Gamma(M, E^{*} \otimes \mathrm{Dens})\right)^{\vee}}$$

top^{l} dual.

Rmk: Have embedding

$$\Gamma(M, E) \hookrightarrow \overline{\Gamma}(M, E).$$

Ex: Sps $E = \mathrm{Dens}$. Have 'Dirac' distribution

$$\delta_p \in \bar{\Gamma}(M, \mathrm{Dens})$$

defined by:

$$\delta_p : f \in C^\infty(M) \longmapsto f(p).$$

- If E_1, E_2 v.b.'s then let

$$\begin{array}{ccc} E_1 & & E_2 \\ & \searrow \; \swarrow & \\ & M & \\ p_1 & \uparrow\uparrow & p_2 \\ & M \times M & \end{array} \qquad \begin{array}{c} E_1 \boxtimes E_2 \\ \| \text{dfn} \\ p_1^* E_1 \otimes p_2^* E_2 . \end{array}$$

A **kernel** is a section

$$k \in \Gamma\left(M \times M, \left(F \otimes \mathrm{Dens}^{1/2}\right) \boxtimes \left(E^* \otimes \mathrm{Dens}^{1/2}\right)\right)$$

Such a kernel defines an operator

$$K : \bar{\Gamma}\left(M, E \otimes \mathrm{Dens}^{1/2}\right) \longrightarrow \Gamma\left(M, F \otimes \mathrm{Dens}^{1/2}\right)$$

by

$$(Ks)(x) = \int_{y \in M} k(x,y)\, s(y)$$

If k_1, k_2 are two kernels s.t.

K_1, K_2 are composable

then $K_1 \circ K_2$ is the operator w/ kernel

$$k(x,y) = \int_{z \in M} k_1(x,z)\, k_2(z,y).$$

<u>Thm</u>: [Schwartz]

$$\begin{array}{ccc} \text{kernels} & \xrightarrow{\cong} & \text{bounded linear operators} \\ \| & & \int \big\downarrow \\ \Gamma(M \times M, (F \otimes \mathrm{Den}^{1/2}) \boxtimes (E^* \otimes \mathrm{Den}^{1/2})) & \longrightarrow & \mathrm{Hom}\left(\overline{\Gamma}(E \otimes \mathrm{Dens}^{1/2}), \Gamma(F \otimes \mathrm{Dens}^{1/2})\right). \end{array}$$

• Dirac notation: If P is any bounded linear operator then we denote

$$p(x,y) = \langle x \,|\, P \,|\, y \rangle$$

its associated kernel. Thus

$$(P s)(x) = \int_{y \in H} \langle x \,|\, P \,|\, y \rangle \, s(y).$$

We are interested in the (family of) operator:

$$e^{-tH}, \quad t > 0.$$

where H = generalized Laplacian.

Dfn: Let H be generalized Laplacian on $E \otimes \mathrm{Dens}^{1/2}$.
A **heat kernel** for H is

$$p_t(x,y) \in \Gamma_{cts}\left(\mathbb{R}_+ \times M \times M, (E \otimes \mathrm{Dens}^{1/2}) \otimes (E^* \otimes \mathrm{Dens}^{1/2})\right)$$

s.t.

1. $p_t(x,y)$ is C^1 wrt t.

2. $p_t(x,y)$ is C^2 wrt $\underline{x}$, for any coordinate $\underline{x}$ of M.

3. $(\partial_t + H_x)\, p_t(x,y) = 0$.

4. $\lim_{t \to 0} P_t s = s$.

in other words

$$\lim_{t \to 0} p_t(x,y) = \delta(x-y).$$

We show such a heat kernel is unique.

Lemma: Assume the formal adjoint H^* has a heat kernel, p_t^*. If

$$s_t : \mathbb{R}_+ \to \Gamma(E \otimes \mathrm{Dens}^{1/2})$$

is C^1 in t, C^2 in x

$$\lim_{t \to 0} s_t = 0, \quad (\partial_t + H)s_t = 0$$

$\Rightarrow s_t = 0$.

Pf: If $s_1(t,-)$, $s_2(t,-)$ are C^2 sections of $E \otimes \mathrm{Dens}^{1/2}$ then

$$\int_M \langle Hs_1(t,-), s_2(t,-) \rangle$$

$$= \int_M \langle s_1(t,-), Hs_2(t,-) \rangle.$$

$\forall\, t > 0$.

For $u \in \Gamma(E^* \otimes \mathrm{Dens}^{1/2})$ define

$$f_u(\theta) = \int_{(x,y) \in M \times M} \langle s(\theta, x), p^*_{t-\theta}(x,y)\, u(y) \rangle.$$

for $0 < \theta < t$.

$$\frac{\partial}{\partial \theta} f_u(\theta) = \int_{(x,y)} \langle \partial_\theta s(\theta, x), p^*_{t-\theta}(x,y)\, u(y) \rangle$$

$$+ \int_{(x,y)} \langle s(\theta, x), H^* p^*_{t-\theta}(x,y)\, u(y) \rangle.$$

$$= \int_{(x,y)} \langle (\partial_\theta + H)\, s(\theta, x), p^*_{t-\theta}(x,y)\, u(y) \rangle$$

$= 0$ by Heat eqn.

$\Rightarrow f_u(\theta)$ is constant.

$$\lim_{\theta \to t} f_u(\theta) = \int_M \langle s(\theta, x), u(x) \rangle$$

But $\lim_{\theta \to 0} f_u(\theta) = 0$. $\Rightarrow$

$$\int_M s(t,x)\, u(x) = 0 \quad \forall \quad t > 0.$$

$\forall\ u \in \Gamma(E^* \otimes \mathrm{Dens}^{1/2})$. $\Rightarrow$ $s(t,-) = 0$.

$\boxtimes$

Cor: 1) Sps $\exists$ p_t^* for H^*, then $\exists$ p_t for H.

2) Sps $\exists$ p_t^*, p_t for H^*, H. Then

$$p_t(x,y) = \left(p_t(y,x)\right)^*.$$

3) Sps $\exists$ p_t^*, p_t for H^*, H. Then the operators $P_t s = \int_M p_t s$ form a semi-group.

Pf: Let

$$f_u(\theta) = \int_M \left((P_\theta s)(x), (P^*_{t-\theta} u)(x) \right)$$

As before, f is constant $\Rightarrow$

$$(P_t s, u) = (s, P^*_t u)$$

This proves 1), 2).

Semigroup $P_{t+s} = P_t P_s$.

$\lim_{t \to 0} s_t = P_\theta s \Rightarrow s_t = P_{t+\theta} s$

by lemma. $\boxtimes$