February 22

"Formal" solution to heat eqn.

To construct a heat kernel for H we assumed the existence of a "approximate" soln. to the heat eqn; call $k_t^N(x,y)$, which exists $\forall\ N > 0$.

Key properties:

① $r_t^N(x,y) \overset{\text{def}}{=} (\partial_t + H_x)\, k_t^N(x,y)$ satisfies

$$\|r_t\|_\ell \leq C(\ell)\, t^{N - \frac{n}{2} - \frac{\ell}{2}}.$$

② $\lim_{t \to 0} K_t^N s = s$.

To construct this approximate soln, we first construct a "formal soln" of the form

$$k_t(x,y) = q_t(x,y) \sum_{i \geq 0} t^i \Phi_i(x,y) \, |dy|^{1/2}.$$

where

① $\Phi_i(x,y) \in \Gamma(M \times M, E \boxtimes E^*)$

are smooth sections, defined in a neighborhood of diag $\subset M \times M$.

② $q_t(x,y)$ is modeled off of the Euclidean heat kernel. In normal coordinates around y:

$$q_t(x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-\|x-y\|^2/4t} \, |d\underline{x}|^{1/2}.$$

• We need to return to some properties of normal coordinates.

For $x_0 \in M$, have exponential

$$\exp_{x_0} : T_{x_0} M \longrightarrow M$$

and we will write $x = \exp_{x_0} \underline{x}$.

Let $j(\underline{x})$ be the determinant of the Jacobian of this coordinate. So:

$$\exp_{x_0}^* dx = j(\underline{x})\, d\underline{x}.$$

volume form

That is:

$$j(\underline{x}) = \left| \det\left(d_{\underline{x}} \exp_{x_0} \right) \right|$$

$$= \sqrt{\det g(\underline{x})}.$$

Let $Eu = \underline{x}^i \frac{\partial}{\partial \underline{x}^i}$.

Prop: ① $\nabla\left(f \, |d\underline{x}|^{1/2}\right)$

$$= \left(df - \frac{1}{2} f \, d \log j\right) |d\underline{x}|^{1/2}.$$

② $\Delta\left(f \, |d\underline{x}|^{1/2}\right) = \left(\left(j^{1/2} \circ \Delta \circ j^{-1/2}\right) f\right) |d\underline{x}|^{1/2}.$

③ $\Delta\left(\|x\|^2\right) = -2\left(n + Eu \cdot \log j\right).$

④ Let $q_t(\underline{x}) = \frac{1}{(4\pi t)^{n/2}} e^{-\|\underline{x}\|^2/4t} \, |d\underline{x}|^{1/2}.$

Then

$$\left(\partial_t + \Delta - j^{1/2}\left(\Delta j^{-1/2}\right)\right) q_t = 0.$$

Pf: ①, ② follow from

$$\nabla\left(|dx|^{1/2}\right) = 0 \Rightarrow \nabla\left(j^{1/2} |d\underline{x}|^{1/2}\right) = 0.$$

(3) Let $\phi \in C^\infty_c(T_{x_0}M)$, then

$$\int (\phi, \Delta \|x\|^2)\, j(\underline{x})\, d\underline{x}$$

$$= \int (d\phi, d\|x\|^2)\, j(\underline{x})\, d\underline{x}$$

Lem: $(Eu, \partial_i) = \underline{x}^i$.   )

Now, $(Eu, \partial_i) = \sum_j \underline{x}^i (\partial_i, \partial_j)$.

$$= \underline{x}^i + O(\|x\|^2).$$

But

$$Eu\,(Eu, \partial_i) = (\underline{\underline{\nabla_{Eu} \overbrace{Eu}^{Eu}}}, \partial_i) + (Eu, \underbrace{\nabla_{Eu}\partial_i})$$

Torsion-free $\Rightarrow$

$$(Eu, \nabla_{Eu}\partial_i) = (Eu, \nabla_{\partial_i} Eu) + (Eu, [Eu, \partial_i])$$

$$= \frac{1}{2}\partial_i \|Eu\|^2 - \underline{\underline{(Eu, \partial_i)}}$$

So $\quad Eu(Eu, \partial_i) = \frac{1}{2}\partial_i \|Eu\|^2 = \underline{x}^i$.

Thus $\quad (Eu, \partial_i) = \underline{x}^i$.

So $\quad (d\phi, d\|\underline{x}\|^2) = (\partial_i \phi \, d\underline{x}^i, 2\underline{x}^j d\underline{x}^j)$

$$= 2\,\partial_i \phi \, (\partial_i, \underline{x}^j \partial_j)$$

$$= 2\,\partial_i \phi \, (\partial_i, Eu)$$

$$= 2\,\underline{x}^i \partial_i \phi = 2\, Eu\, \phi.$$

$\Rightarrow$

$$\int (\phi, \Delta \|\underline{x}\|^2)\, j(\underline{x})\, d\underline{x} = 2\int (Eu\,\phi)\, j(\underline{x})\, d\underline{x}.$$

$$= -2\int (n + Eu(\log j))\, \phi(\underline{x})\, j(\underline{x})\, d\underline{x}.$$

④ Leibniz:

$$\Delta\left(e^{-\|\underline{x}\|^2/4t}\,|d\underline{x}|^{1/2}\right)$$

$$= \Delta\left(e^{-\|\underline{x}\|^2/4t}\right)|d\underline{x}|^{1/2}$$

$$+\ t^{-1}\,\underline{x}^i\,e^{-\|x\|^2/4t}\,\nabla_{\partial_i}|d\underline{x}|^{1/2}.$$

$$+\ e^{-\|x\|^2/4t}\,\Delta|dx|^{1/2}.$$

$$= \Delta\left(e^{-\|\underline{x}\|^2/4t}\right)|d\underline{x}|^{1/2}$$

$$+\ t^{-1}\,e^{-\|x\|/4t}\,\nabla_{Eu}|d\underline{x}|^{1/2}$$

$$+\ e^{-\|x\|^2/4t}\,\Delta|d\underline{x}|^{1/2}.$$

$$\Delta e^{-f} = -\partial_i \partial_i e^{-f}$$

$$= -\partial_i \left(-\partial_i f \, e^{-f} \right)$$

$$= \left(-\Delta f + (\partial_i f)(\partial_i f) \right) e^{-f} .$$

$\Rightarrow$

$$\Delta \left(e^{-\|\underline{x}\|^2/4t} \right) = \left(2\left(n + Eu(\log j)\right) - \frac{\|\underline{x}\|^2}{t} \right) \frac{e^{-\|\underline{x}\|^2/4t}}{4t} .$$

$\rightsquigarrow$

$$\left(\partial_t + \Delta - j^{1/2} \Delta \left(j^{1/2} \right) \right) q_t = 0 .$$