February 27:

G = Lie group. M = smooth manifold.

A principal G-bundle on M is

1) P = manifold, $\begin{array}{c} P \\ \downarrow \pi \\ M \end{array}$ smooth surjective map.

2) A smooth G-action on P

$$G \times P \longrightarrow P, \quad (g, p) \longmapsto pg^{-1}.$$

such that $\pi(pg^{-1}) = \pi(p)$, and s.t. G acts freely and transitively on each $\pi^{-1}(p) \subset P$.

This data must be s.t. $\exists$ a nbd U about each pt $x \in M$ and a smooth map

$$h : \pi^{-1}(U) = P|_U \longrightarrow G$$

s.t.

1) $h(p \cdot g^{-1}) = h(p) g^{-1}$. ($G$-equivariant)

2) $\varphi = (\pi, h) : \pi^{-1}(U) \longrightarrow U \times G$

is a diffeomorphism.

Equivalent definition: Čech cocycle.

Sps $\mathcal{U} = \{U_\alpha\}$ is an open cover for M. And

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G \quad , \text{ smooth}$$

are s.t:

1) $g_{\alpha\alpha}(x) = 1 \in G$ for all $x \in U_\alpha$.

2) $g_{\alpha\beta}^{-1} = g_{\beta\alpha}$.

3) $g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha}(x) = 1$ for all

$x \in U_\alpha \cap U_\beta \cap U_\gamma$.

Call this "G-cocycle" data. Two such data are equivalent if $\exists\ h_\alpha : U_\alpha \to G$ such that

$$h_\alpha\big|_{U_\beta}\, g_{\alpha\beta} = g'_{\alpha\beta}\, h_\beta\big|_{U_\alpha}.$$

Fact: $\{\text{Principal } G\text{-bundle}\}/\simeq \ \cong\ \{G\text{-cocycle data}\}/\simeq$.

Classifying spaces :

Given $f : M \longrightarrow N$, $\begin{array}{c} P \\ \downarrow \pi \\ N \end{array}$ a principal G bundle on N

$\rightsquigarrow$ $f^* P$ is a principal G bundle on M

$$\begin{tikzcd} f^*P \arrow[d, equal] & \\ \{ (x, p) \mid p \in \pi^{-1}(f(x)) \} \arrow[r] \arrow[d, "f^*\pi"] & P \arrow[d, "\pi"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

<u>Prop</u> : If $f, g : M \to N$ are homotopic,

then $f^* P \cong g^* P$.

<u>Pf (Sketch)</u>: Sps $F : [0,1] \times M \longrightarrow N$ is a homotopy $f \underset{F}{\simeq} g$. Then consider $\begin{array}{c} F^* P \\ \downarrow \\ [0,1] \times M \end{array}$.

This bundle has the property that

$$F^* P \big|_{0 \times M} \cong f^* P \quad , \quad F^* P \big|_{1 \times M} \cong g^* P$$

Since $[0,1]$ is contractible it is a consequence that $f^*P\big|_0 \cong f^*P\big|_1$. $\square$

As a corollary, if $B \simeq *$, then any principal G-bundle over B is trivial. Can use this to prove:

<u>Thm</u>: Suppose $\begin{matrix} EG \\ \downarrow \\ BG \end{matrix}$ is a principal G-bundle w/ EG (weakly) contractible. Then there is a bijection

$$[X, BG] \xrightarrow[\Phi]{\cong} \left\{ \begin{matrix} \text{Principal } G\text{-bundles} \\ \text{on } X \end{matrix} \right\} \Big/ \cong$$

$$\Phi(f) = f^* EG.$$

<u>Theorem</u> [Milnor] $\begin{matrix} EG \\ \downarrow \\ BG \end{matrix}$ exists. (In fact, for any topological group.)

- Principal bundles make sense even when G is <u>discrete</u>. Then, a G-bundle is simply a $\#G$-sheeted covering space w/ G as the group of deck transformations.

In this case we can use the homotopy LES for

$$\begin{array}{c} EG \simeq * \\ \downarrow \\ BG \end{array}$$

to see that $BG = K(G,1)$. In particular

$$\left\{ \begin{array}{c} G\text{-covering} \\ \text{spaces} \\ \downarrow \\ X \end{array} \right\} \Big/ \simeq \quad \xleftrightarrow{\simeq} \quad [X, BG]$$

$$\| \qquad\qquad \text{singular cohomology}$$

$$[X, K(G,1)] = H^1(X; G).$$

<u>Examples</u>: • On $S^2 = \mathbb{P}^1 = N \cup S$ we have the conn $\rightsquigarrow$ Principal $U(1)$ bundle for each $m \in \mathbb{Z}$.

$$N \cap S \xrightarrow{\sim} S^1 = U(1) \xrightarrow[(-)^m]{} U(1)$$

$$g_m$$

Ex: $SU(2)$ bundles over S^4.

Again look at $S^4 = N \cup S$. Then

$$\begin{tikzcd} N \cap S \simeq S^3 \simeq SU(2) \arrow[r, "(-)^m"] \arrow[r, bend right, "g_m"'] & SU(2) \end{tikzcd}$$

For each m g_m defines a principal $SU(2)$ bundle over S^4, call it P_m.

$$\begin{tikzcd} P_1 \cong S^7 \arrow[d] \\ S^4 \end{tikzcd}$$

diffeomorphic.

$\circlearrowleft SU(2) \subset \mathbb{H}$ acts $g \cdot (p,q) = (pg^{-1}, qg^{-1})$.

[Present $S^7 = \{ (p,q) \in \mathbb{H} \times \mathbb{H} \mid |p|^2 + |q|^2 = 1 \}$

Also, recall $S^3 = SU(2) \subset \mathbb{H}$ is the group of unit quaternions.

To see S^4 look at the map:

$$\begin{tikzcd} S^7 \arrow[r] \arrow[dr, dashed, "\pi"'] & \mathbb{R}^5 \\ & S^4 \arrow[u, phantom, sloped, "\subset"] \end{tikzcd}, \quad (p,q) \overset{\pi}{\longmapsto} (2p\bar{q}, |p|^2 - |q|^2).$$

unit length.

Note that π is constant along the $SU(2)$ orbits. This is the bundle projection.

<u>Ex</u>: $H < G$ closed Lie subgroup.

$$\begin{tikzcd} G \arrow[d] \\ G/H \end{tikzcd}$$ is a principal H-bundle.

<u>Ex</u>: $\mathbb{C}P^n$ has cover $\mathcal{U} = \{U_0, \dots, U_n\}$ s.t.

$U_i \cong \mathbb{C}^n$. $\quad U_i = \{z_i \neq 0\}$.

$$g_{ij} : U_i \cap U_j \longrightarrow U(1)$$

$$[z_0 : \dots : z_n] \longmapsto \frac{|z_j|}{z_j} \cdot \frac{z_i}{|z_i|}.$$

$\rightsquigarrow$ Principal $U(1)$-bundle $\begin{tikzcd} P \arrow[d] \\ \mathbb{C}P^n \end{tikzcd} \cong S^{2n+1}$.

<u>Back to geometry</u>: Sps that $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ is a vector bundle of rank r (defined over a field k). There is a bundle Fr_E on M whose fiber over $x \in M$ is $GL(E_x) \doteq GL(r)$. There is the natural structure of a principal $GL(r)$-bundle on Fr_E. Called the <u>bundle</u> of <u>frames</u>.

- Let $\begin{matrix} Fr_M \\ \downarrow GL(n) \\ M \end{matrix}$ denote Fr_{T_M} = bundle of linear frames

$n = \dim M, \quad k = \mathbb{R}$.

- A <u>Riemannian</u> <u>str</u> on $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ allows us to

define a $O(r)$-bundle of orthonormal frames of E

$E = T_M$

$\rightsquigarrow$ $\begin{matrix} Fr_M^O \\ \downarrow O(n) \\ M \end{matrix}$ = bundle of orthonormal frames. on Riem. manifold (M, g)

- Consider, on a Riem. vector bundle E:

$$\begin{matrix} Fr_E^O / SO(r) & = & \text{two-sheeted covering of } M \\ \downarrow & & \text{12} \\ w_1(E) & \in & H^1(M; \mathbb{Z}) \end{matrix}$$

1st Stiefel-Whitney class. $\rightsquigarrow$

Prop: E is orientable iff $w_1(E) = 0$.

Pf: E is orientable $\Leftrightarrow$ $Fr_E^O / SO(r)$ is trivial. ▨

An orientation is a choice of section of $Fr_E^O / SO(r)$

$\rightsquigarrow H^0(X; \mathbb{Z}/2)$.

The class $w_1(E)$ is called a characteristic class.

- More generally, if G is any Lie group, a universal characteristic class is an element of

$$c \in H^*(BG; \Lambda), \quad \Lambda = \text{any ring}.$$

Given a class c we can pull-back along a classifying map $f_P : X \longrightarrow BG$

$$P \simeq f_P^* EG.$$

$$\rightsquigarrow f_P^* c \in H^*(X; \Lambda).$$

The most important feature of char classes is naturality:

$$\begin{array}{ccc} & & E \\ & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array} \qquad f^* c_E = c_{f^*E}.$$

This is automatic for classes pulled back from universal char classes.

Eg: $\begin{array}{c} EO(n) \\ \downarrow \\ BO(n) \end{array}$ is the universal $O(n)$-bundle.

$\rightsquigarrow EO(n)/SO(n)$ is the universal orientation bundle

$\rightsquigarrow w_1 = w_1(EO(n)) \in H^1(BO(n); \mathbb{Z}/2).$

is a universal char. class.

In fact

$$H^\cdot(BO(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, \dots, w_n]$$

$$|w_i| = i.$$

We have the diagram:

$$\begin{tikzcd} & BU(n) \arrow[d, "Bi"] \\ M \arrow[ur, "f_V"] \arrow[r, "f_E"'] & BSO(2n) \end{tikzcd}$$

$$c_1(V) = f_V^* c_1$$

$$w_2(V) = f_E^* w_2.$$

$$f_E = Bi \circ f_V.$$

It suffices to show that

$$Bi^* : H^2(BSO(2n); \mathbb{Z}/2) \to H^2(BU(n); \mathbb{Z}/2)$$

is injective. It's actually an isomorphism.

- I suggest doing this differently.