February 4

- We move on to representations.

A representation of and algebra A defined over k is a k-vector space W and a k-algebra homomorphism

$$\rho : A \longrightarrow \mathrm{End}(W).$$

When $A = C\ell(V, q)$ we will write

$$\rho(\varphi).w = \varphi \cdot w$$

for $\varphi \in C\ell(V, q)$, $w \in W$.

- Recall that a complex vector space is a real vector space V together w/ an endomorphism J s.t. $J^2 = -\mathbb{1}$. Similarly, a <u>complex representation</u> is a real representation (W, ρ) s.t.:

$$\rho \circ J = J \circ \rho.$$

Similarly one defines the concept of a <u>quaternionic representation</u>.

- A representation $\rho : A \to \mathrm{End}(W)$ is called <u>reducible</u> if $\exists\ W_1, W_2 \subseteq W$ s.t.

$$W = W_1 \oplus W_2$$

$$\rho = \rho_1 \oplus \rho_2$$

where $\rho_i : A \to \mathrm{End}(W_i)$. Any f.d. rep^n can be decomposed into a sum of irreducible representations

$$\rho = \rho_1 \oplus \cdots \oplus \rho_m .$$

- We will be studying rep^n up to equivalence:

$$\begin{tikzcd}
A \times W \arrow[r, "\rho"] \arrow[d, "\mathbb{1} \times F"'] & W \arrow[d, "F", "\cong"'] \\
A \times W' \arrow[r, "\rho'"'] & W'
\end{tikzcd}$$

Thm: [Matrices are simple]

Let $k = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Then the only irreducible real rep^n of $k(n) = \{n \times n$ matrices valued in $k\}$ is the defining rep^n k^n.

Pf: $k(n)$ is simple. $\blacksquare$

$\rightsquigarrow$ This leads to the observations about the algebra $C\ell_n$:

- If $n+1 \equiv 0 \mod 4$ then the # of inequivalent irreps is $\underline{\underline{2}}$.

- If $n+1 \not\equiv 0 \mod 4$ then the # of inequivalent irreps is $\underline{\underline{1}}$.

Similarly, for the complex Clifford alg $\mathbb{C}l_n$:

- If n odd there are $\underline{\underline{2}}$ irreps.

- If n even there is $\underline{\underline{1}}$ irrep.

<u>Prop</u>: $Cl_{n+1}^{ev} \simeq Cl_n$.

<u>Pf</u>: $\{e_i\}$ o.n.b. for $\mathbb{R}^{n+1}$. Define

$$\mathbb{R}^n \longrightarrow Cl_{n+1}^{0}$$

$$e_i \longmapsto e_i e_{n+1}.$$

Since $(e_i e_{n+1})^2 = -e_i^2 e_{n+1}^2 = -1$ this extends to a homomorphism

$$f : Cl_n \longrightarrow Cl_{n+1}^{ev}.$$

Isomorphism since we hit all generators. $\boxtimes$
and dim agree.

We will use $Cl_{n+1}^{ev} \cong Cl_n$ irreps
$\cup$

to build Spin$(n+1)$ irreps.

Let $\mathcal{M}_n$ = Grothendieck group of Cl_n-mod.

$\cong$ $\left\{ \begin{array}{l} \text{free abelian group on equivalence} \\ \text{classes } [L] \text{ where } L \text{ is irrep} \\ \text{for the algebra } Cl_n \end{array} \right\}$

$\mathcal{M}_n^{\mathbb{C}}$ = same for $\mathbb{C}l_n$.

- $Cl_1 \cong \mathbb{C}$. There is only one irrep, it is $\mathbb{C}$. This is real 2-dimensional.

$$\mathcal{M}_1 \cong \mathbb{Z}.$$

- $Cl_2 \cong \mathbb{H}$. Only one irrep, it is $\mathbb{H}$. This is real 4-dim^l.

$$\mathcal{M}_2 \cong \mathbb{Z}.$$

- $Cl_3 \cong \mathbb{H} \oplus \mathbb{H}$. There are two irreps $\mathbb{H}_1, \mathbb{H}_2$ each 4-dim^l.

$$\mathcal{M}_3 \cong \mathbb{Z} \oplus \mathbb{Z}.$$

- $Cl_4 \cong \mathbb{H}(2)$ There is one irrep $\mathbb{H}^2$, it is 8-dim^l.

$\vdots$

$$\mathcal{M}_{n+4} \cong \mathcal{M}_n.$$

- $\mathbb{C}l_1 \cong \mathbb{C} \oplus \mathbb{C}$. There are 2 irreps $\mathbb{C}_1, \mathbb{C}_2$. Each $1_{\mathbb{C}}$-dim^l.

$$\mathcal{M}_1^{\mathbb{C}} \cong \mathbb{Z} \oplus \mathbb{Z}.$$

- $\mathbb{C}l_2 \cong \mathbb{C}(2)$. there is one irrep $\mathbb{C}^2$, it is $2_{\mathbb{C}}$-dim^l.

$$\mathcal{M}_2^{\mathbb{C}} \cong \mathbb{Z}.$$

$\vdots$

$$\mathcal{M}_{n+2}^{\mathbb{C}} \cong \mathcal{M}_n^{\mathbb{C}}.$$

Recall the volume element

$$\omega = e_1 \cdots e_n \in Cl_n$$

Define $\omega_{\mathbb{C}} = i^{(n+1)/2} \omega \in \mathbb{C}l_n$.

Not even/odd !!!

- Case $n \equiv 3 \mod 4$. Then

$$\omega^2 = 1 \Rightarrow Cl_n = Cl_n^+ \oplus Cl_n^-$$

$$Cl_n^\pm = (\mathbb{1} \pm \omega) Cl_n .$$

<u>Prop</u>: Let (W, ρ) be Cl_n-irrep, $n \equiv 3 \mod 4$.
Then

$$\rho(\omega) = \pm \mathbb{1} .$$

<u>Pf</u>: Since $\rho(\omega^2) = \mathbb{1}$, we have decomposition

$$W = W^+ \oplus W^- .$$

Since ω is in the center of Cl_n we see that W^+, W^- are each Cl_n-inv't. So, one has to be trivial. $\boxtimes$

Now, $n+1 \equiv 0 \mod 4$. So if W is a Cl_{n+1}-rep we have seen how

$$W = W^+ \oplus W^-$$

where $W^{\pm} = (\mathbb{1} \pm \rho(\omega_{n+1}))\, W$.

Since ω_{n+1} commutes w/ Cl_{n+1}^{ev}, we see that each $W^{\pm}$ is a representation for $Cl_{n+1}^{ev} \cong Cl_n$. These are the $W^{\pm}$ from the previous proposition.

- We now construct representations for the Lie group $Spin(n)$.

A <u>spinor rep^n</u> of $\mathrm{Spin}(n)$ is an irrep summand of the restriction of a $\mathcal{C}\ell(n)$-irrep along

$$\mathrm{Spin}(n) \hookrightarrow \mathcal{C}\ell_n^{ev} \hookrightarrow \mathcal{C}\ell_n .$$

Suppose $n \equiv 3 \bmod 4$. The <u>spinor rep^n</u> of $\mathrm{Spin}(n)$ is Δ_n as:

$$\begin{tikzcd}
\mathrm{Spin}(n) \arrow[r, hook] \arrow[drr, dashed, "\Delta_n"'] & \mathcal{C}\ell_n \arrow[r, "\rho"] & \mathrm{End}(S) \\
& & GL(S) \arrow[u, hook]
\end{tikzcd}$$

where ρ is any $\mathcal{C}\ell_n$ irrep.

<u>Claim</u>: This is well-defined. (There is a unique spinor rep when $n \equiv 3 \bmod 4$).

<u>Pf</u>: Since $\mathcal{M}_n = \mathbb{Z} \oplus \mathbb{Z}$, there is something to check here. Recall the automorphism

$$\alpha : \mathcal{C}\ell_n \circlearrowleft$$

which witnesses $Cl_n = Cl_n^{ev} \oplus Cl_n^{odd}$.

When $n \equiv 3 \mod 4$ we have $\alpha(\omega) = -\omega$

$\Rightarrow \alpha : Cl_n^{\pm} \to Cl_n^{\mp}$.

$\Rightarrow Cl_n^{ev} = \{ (\varphi, \alpha(\varphi)) \mid \varphi \in Cl_{ev}^{+} \}$.

So the restriction of the two irreps of Cl_n are equivalent when restricted to Cl_n^{ev}.

- Similarly, if $n \equiv 5, 6 \mod 8$ one sees from the classification that the restriction of a Cl_n-irrep remains an irrep when restricted to $Cl_n^{ev} \cong Cl_{n-1}$.

- When $n \equiv 1, 2 \mod 8$ then the restriction of an irrep to $Cl_n^{ev} \cong Cl_{n-1}$ is a sum of two copies of an irrep.

- When $n \equiv 0 \bmod 4$ then the restriction splits into two inequivalent irreps.

Summary:

$n \equiv 0 \bmod 8$: There is a unique Cl_n irrep. call it W. When we restrict W to $Cl_n^{ev} \cong Cl_{n-1}$ it decomposes

$$W = W^+ \oplus W^-$$

where ω_{n-1} acts by ± 1 on $W^\pm$.

There are thus two irreps of $\mathrm{Spin}(n)$

$$S^\pm = W^\pm$$

- $n \equiv 1 \mod 8$: There is a unique

Cl_n irrep call it W. Sps $n = 8k+1$.

Then
$$\dim_{\mathbb{R}} W = 2^{4k+1}.$$

On the other hand, the dim of the unique $Cl_n^{ev} = Cl_{n-1}$ irrep W' is 2^{4k}. So:

$$W = W' \oplus W'.$$

There is a unique Spin(n) irrep

$$S = W'.$$

- $n \equiv 2 \mod 8$: There is a unique

Cl_n-irrep W. Again its restriction to $Cl_n^{ev} \cong Cl_{n-1}$ is $W = W' \oplus W'$

where W' is the unique irrep of Cl_{n-1}.

There is a unique irrep call it
$$S = W'.$$

By defn, note that S is a <u>complex</u> rep^n.

- <u>$n \equiv 3 \mod 8$</u>: There is a unique $\mathrm{Spin}(n)$ irrep S. It is the restriction of either $W^\pm$ where ω acts by ± 1.

By dfn, S is <u>quaternionic</u>.

- <u>$n \equiv 4 \mod 8$</u>: There is a unique Cl_n-irrep. Its restriction to $Cl_n^{ev} \cong Cl_{n-1}$ is a sum $W = W^+ \oplus W^-$.

The two $\mathrm{Spin}(n)$ irreps are
$$S^\pm = W^\pm.$$

Each $S^\pm$ are <u>quaternionic</u>.

- $n \equiv 5 \mod 8$: There is unique $\mathrm{Spin}(n)$ irrep S. S is quaternionic.
- $n \equiv 6 \mod 8$: There is a unique $\mathrm{Spin}(n)$ irrep S. S is complex.
- $n \equiv 7 \mod 8$: There is a unique $\mathrm{Spin}(n)$ imp. S.

The complex classification is even easier.

*: Note, we are talking about complex spin representations for the ordinary spin group $\mathrm{Spin}(n) \subset C\ell^0_n$.

A complex spin rep is the restriction of a $\mathbb{C}l_n$ irrep along:

$$\mathrm{Spin}(n) \subset Cl_n^{ev} \subset \mathbb{C}l_n.$$

Let's start w/ the case that $n = 2m$ is even. So: $\mathbb{C}l_{2m}$ is the $\mathbb{C}$-clifford algebra associated to $V = \mathbb{C}^{2m}$ w/ its nondeg quadratic form. By nondegeneracy

$$V = L \oplus L^* \ni (v, f).$$

where $q(l, \psi) = \psi(l) = \langle l, \psi \rangle$.

Let $S \overset{\text{def}}{=} \Lambda^{\cdot} L^* = \mathbb{C} \oplus L^* \oplus \cdots \oplus \Lambda^m L^*$.

We will construct an isomorphism

$$\mathbb{C}l_n \cong \mathrm{End}(S).$$

Define

$$\begin{array}{ccc} L & \longrightarrow & \mathrm{End}(S) \\ \psi & & \psi \\ l & \longmapsto & i_l = \langle l, - \rangle . \end{array}$$

"contract"

$$\begin{array}{ccc} L^* & \longrightarrow & \mathrm{End}(S) \\ \psi & & \psi \\ \psi & \longmapsto & \psi \wedge (-) \end{array}$$

"wedge".

<u>Claim</u>: For $l \in L$, $\psi \in L^*$ one has

$$(i_l + \psi)^2 = 2\langle l, \psi \rangle \cdot \mathbb{1}.$$

Thus we obtain an alg homomorphism

$$\mathbb{C}l_{2m} \cong Cl(L \oplus L^*) \xrightarrow[\rho_{\mathbb{C}}]{} \mathrm{End}(S).$$

To see its an isomorphism it suffices to check the case $m = 1$. Then

$$S = \Lambda^\bullet \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}.$$

Then $\mathbb{C}l_2 \cong \mathbb{C}(2)$. Check that

$$\begin{tikzcd} \mathbb{C}l_2 \arrow[r, "\rho_{\mathbb{C}}"] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{End}(S) \arrow[d, phantom, "\cong" rotate=90] \\ \mathbb{C}(2) \arrow[r, dashed, "\cong"] & \mathrm{End}(\mathbb{C}^2) \end{tikzcd}$$

<u>So</u>: when $n = 2m$ have $\mathbb{C}l_{2m} \cong \mathrm{End}(S)$

where $S = \Lambda^\bullet L^* = \Lambda^\bullet \mathbb{C}^m$. Note that

S is equipped with

$$S = \underbrace{S^+}_{\Lambda^{ev} L^*} \oplus \underbrace{S^-}_{\Lambda^{odd} L^*}.$$

It is clear that this decomposition is compatible w/ the parity decomposition for $\mathbb{C}l(V) = \mathbb{C}l^{ev} \oplus \mathbb{C}l^{odd}$.

w/ $S^{\pm}_{\mathbb{C}}$ are each rep^n for $\mathbb{C}l^{even}$.

<u>Dfn</u>: Let $n = 2m$ be even. The <u>$\pm$ complex spin reps</u> of $Spin(2m)$ are:

$$Spin(2m) \hookrightarrow \mathbb{C}l^{ev}_{2m} \xrightarrow{\rho^{\pm}_{\mathbb{C}}} End(S^{\pm}_{\mathbb{C}})$$

In particular $S^{\pm} = \Lambda^{ev/odd}(\mathbb{C}^m)$.

$$\dim_{\mathbb{C}} S^{\pm} = 2^{m-1}.$$

- Next, for odd dimension, $n = 2m+1$.

Since

$$\mathbb{C}l_{2m+1} \cong \mathbb{C}l_{2m} \oplus \mathbb{C}l_{2m}$$

there are <u>two</u> irreps for $\mathbb{C}l_{2m+1}$.

But, there restrictions to $\mathbb{C}l^{ev}_{2m+1} \cong \mathbb{C}l_{2m}$ agree.

<u>Dfn</u>: Let $n = 2m+1$. The complex spin rep of $Spin(2m+1)$ is the composition

$$Spin(2m+1) \hookrightarrow \mathbb{C}l^{ev}_{2m+1} \longrightarrow End(S)$$

where S is any irrep for $\mathbb{C}l_{2m+1}$.
In particular

$$\dim_{\mathbb{C}} S = 2^m.$$