February 6

Dfn: 1) Let E_1, E_2 be v.b.'s and let $D : \Gamma(M, E_1) \to \Gamma(M, E_2)$ be a differential operator. The *formal adjoint*

$$D^a : \Gamma(M, E_2^a \otimes \mathrm{Dens}_M) \to \Gamma(M, E_1^a \otimes \mathrm{Dens}_M)$$

is:

$$\int_M \langle D\alpha, \beta \rangle = \int_M \langle \alpha, D^a \beta \rangle,$$

where $\alpha \in \Gamma_c(M, E_1)$, $\beta \in \Gamma(M, E_2^a \otimes \mathrm{Dens}_M)$.

2) If E is Hermitian bundle and

$$D \in \mathcal{D}(M, E \otimes \mathrm{Dens}^{1/2})$$

we say D is *symmetric* if

$$D = D^*.$$

- For Riemannian manifolds we have a canonical nowhere vanishing section $|dx|$ of Dens_M.

$\Rightarrow \quad \Gamma(M, E) \xrightarrow{\cong} \Gamma(M, E \otimes \mathrm{Dens}_M)$.

Thus, in the Riemannian setting, the formal adjoint of $D : \Gamma(M, E_1) \to \Gamma(M, E_2)$ is

$$D^a : \Gamma(M, E_2^a) \longrightarrow \Gamma(M, E_1^a).$$

- M Riemannian. The formal adjoint of

$$d : \Omega^\cdot(M) \longrightarrow \Omega^{\cdot+1}(M)$$

is an operator

$$\begin{array}{ccc} \Gamma(\Lambda^{\cdot+1} T_M \otimes \mathrm{Dens}_M) & \longrightarrow & \Gamma(\Lambda^\cdot T_M \otimes \mathrm{Dens}_M) \\ g \ \wr\!\wr & & \wr\!\wr \ g . \\ \Omega^{\cdot+1}(M) & \xrightarrow{d^a} & \Omega^\cdot(M) \end{array}$$

which satisfies

$$\int_M (d\alpha, \beta)\,|dx| = \int_M (\alpha, d^*\beta)\,|dx| .$$

In particular note that

$$\int_M d^*\beta \quad |dx| = 0$$

called the "divergence"

for all $\beta \in \Omega^1(M)$.

local expression

$$d^*(\beta_i dx^i) = g^{ij}\partial_i \beta_j .$$

- If $\alpha \in \Omega^1(M)$ then:

$$\begin{tikzcd}
\nabla\alpha \arrow[d, mapsto] & \in & \Gamma(M, T^*_M \otimes T^*_M) \arrow[d, "Tr"] \\
Tr(\nabla\alpha) & \in & C^\infty(M)
\end{tikzcd}$$

In local orthonormal frame

$$\mathrm{Tr}(\nabla\alpha) = \sum_i e_i\,\alpha(e_i) - \alpha(\nabla_{e_i} e_i).$$

<u>Prop</u>:

$$d^2\alpha = -\mathrm{Tr}(\nabla\alpha).$$

<u>Pf</u>: $X \in \mathrm{Vect}(M) \rightsquigarrow$

$$\nabla X \in \Gamma(M, T^*_M \otimes T_M) \cong \Gamma(M, \mathrm{End}\, T_M).$$

This is the endomorphism $(\nabla X)\cdot Y = \nabla_Y X$.

<u>Lemma</u>: $L_X |dx| = \mathrm{Tr}(\nabla X)\, |dx|$.

Given this lemma, we proceed with the proof. For $f \in C^\infty(M) \rightsquigarrow$

$$L_X(f\,|dx|) = (Xf)\,|dx| + f\, L_X |dx|$$

$\Rightarrow$ Since $\int L_X(f|dx|) = 0$:

$\int_M (\text{total derivative}) = 0.$

$$0 = \int_M L_X(f|dx|)$$

$$= \int_M (Xf)|dx| + \int_M \mathrm{Tr}(\nabla X)\,|dx|$$

Now, sps $\alpha = X^{\#}$, the one-form corresponding to X via the metric. Then

$$Xf = (\alpha, df)$$

$\Rightarrow$

$$\int_M (\alpha, df)\,|dx| = -\int_M \mathrm{Tr}(\nabla X) f\,|dx|$$

$$= -\int_M \mathrm{Tr}(\nabla \alpha) f\,|dx|.$$

$\Rightarrow$ $$d^{*}\alpha = -\mathrm{Tr}(\nabla\alpha). \qquad \blacksquare$$

Now to prove the lemma. Let $\{e_i\}$ be local orthonormal frame, and $\{\theta^i\}$ dual frame. Then

$$|dx| = \theta^1 \wedge \cdots \wedge \theta^n.$$

So, for any v.f. X have

$$L_X |dx| = d\, i_X |dx|$$

$$= \sum_j \theta^1 \wedge \cdots \wedge L_X \theta^j \wedge \cdots \wedge \theta^n.$$

Now

$$(L_X \theta^j)(Y) = X(\theta^j(Y)) - \theta^j(L_X Y)$$

$$= X(g(e_j, Y)) - g(e_j, [X,Y])$$

$$= g(\nabla_X Y - [X,Y], e_j) + g(Y, \nabla_X e_j)$$

$$= \theta^j(\nabla_Y X) + g(Y, \nabla_X e_j).$$

Since we have o.n.f $(\nabla_X e_j, e_j) = 0$
$\Rightarrow$

$$g(Y, \nabla_X e_j) \in \operatorname{span}\{\theta^k \mid j \neq k\}.$$

$\Rightarrow$

$$L_X(|dz|) = \sum_j \theta^1 \wedge \cdots \wedge (\theta^j \circ \nabla X) \wedge \cdots \wedge \theta^n$$

Evaluating against $e_1 \wedge \cdots \wedge e_n \Rightarrow$

$$\sum_j \theta^j(\nabla_{e_j} X) = \operatorname{Tr} \nabla X. \quad \boxtimes$$

Similarly:

<u>Prop</u>: Let

$$\iota : \Gamma(T^0_M \otimes \Lambda^{\cdot} T^*_M) \longrightarrow \Gamma(\Lambda^{\cdot -1} T^*_M)$$

be contraction wrt g. Then

$$d^* = -\iota \circ \nabla$$

acting on $\Omega^{\cdot}(M)$.

We characterize the formal adjoint of the Laplacian Δ^E associated to a connection (E, ∇) on (M, g).

For any s, Dens^s_M has a canonical connection, and it preserves the canonical section $|dx|^s$. That is:

$$\Delta^{E \otimes \mathrm{Dens}^s}(\phi |dx|^s) = (\Delta^E \phi) |dx|^s$$

- We contemplate the following Laplacian:

$$\Delta^{E \otimes \mathrm{Dens}^{1/2}} \in \mathcal{D}(M, E \otimes \mathrm{Dens}^{1/2}).$$

Notice the formal adjoint is of the form:

$$(\Delta^{E \otimes \mathrm{Dens}^{1/2}})^* \in \mathcal{D}(M, E^* \otimes \mathrm{Dens}^{1/2}).$$

Next time we will describe this...