February 8

Heat kernel: Define

$$q_t \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$$

by

$$q_t(x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-\|x-y\|^2/4t}.$$

Then if Δ_x = flat Laplacian, we have

$$\Delta_x q_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \Delta\left(e^{-\|x-y\|^2/4t}\right)$$

$$= \frac{1}{(4\pi t)^{n/2}} \partial_i \left(\frac{(x^i - y^i)}{2t} e^{-\|x-y\|^2/4t} \right)$$

$$= \frac{1}{(4\pi t)^{n/2}} \left(\frac{n}{2t} - \frac{\|x-y\|^2}{4t^2} \right) e^{-\|x-y\|^2/4t}.$$

On the other hand

$$\frac{\partial}{\partial t} q_t(x,y) = \frac{1}{(4\pi)^{n/2}} \left(-\frac{n}{2} \frac{1}{t^{1+\frac{n}{2}}} + \frac{\|x-y\|^2}{4t^2} \right) e^{-(\ )}$$

$\Rightarrow$

$$\left(\Delta_x + \frac{\partial}{\partial t} \right) q_t(x,y) = 0 .$$

This is the heat equation.

Lemma:

$$\frac{1}{(4\pi)^{1/2}} \int_{\mathbb{R}} e^{-v^2/4} v^k \, dv \overset{\text{dfn}}{=} a(k)$$

$$= \begin{cases} 0 , & k \text{ odd} \\ \dfrac{k!}{(k/2)!} , & k \text{ even.} \end{cases}$$

Pf: This is "Wick's lemma". Here is a quick proof.

$$A(t) = \sum_{k \geq 0} \frac{t^k}{k!} a(k)$$

$$= \frac{1}{(4\pi)^{1/2}} \int_{\mathbb{R}} \sum_{k \geq 0} \frac{1}{k!} (tv)^k e^{-v^2/4} \, dv$$

$$= \frac{1}{(4\pi)^{1/2}} \int_{\mathbb{R}} e^{-v^2/4 + tv} \, dv.$$

$u = v - 2t$

complete square

$$= \frac{1}{(4\pi)^{1/2}} \int_{\mathbb{R}} e^{-u^2/4 + t^2} \, du$$

$$= \frac{1}{(4\pi)^{1/2}} \cdot (4\pi)^{1/2} \cdot e^{t^2}$$

$$= \sum_{k \geq 0} \frac{t^{2k}}{k!} \Rightarrow a(k) = \begin{cases} k!/(k/2)!, & \text{even} \\ 0, & \text{odd} \end{cases}$$

$\boxtimes$

Define the C^{ℓ}-norm:

$$\|\phi\|_{\ell} = \sup_{k \leq \ell} \sup_{x \in \mathbb{R}} \left| \frac{d^k}{dx^k} \phi(x) \right| .$$

Let

$$(Q_t \phi)(x) = \int_{\mathbb{R}} q_t(x,y) \, \phi(y) \, dy .$$

The following lemma justifies thinking about $q_t(x,y)$ as an integral kernel for the operator:

$$e^{-t\Delta} = \sum_{k \geq 0} \frac{(-t)^k}{k!} \Delta^k .$$

Prop: For ℓ even, $\|\phi\|_{\ell+1} < \infty$:

$$\left\| Q_t \phi - \sum_{k=0}^{\ell/2} \frac{(-t)^k}{k!} \Delta^k \phi \right\| \leq O\left(t^{\ell/2+1}\right) .$$

Pf: Let $y = x + t^{1/2} v$:

$$(Q_t \phi)(x) = \frac{1}{(4\pi)^{1/2}} \int_{\mathbb{R}} e^{-v^2/4} \phi(x + t^{1/2} v)\, dv.$$

Taylor expand, assuming $\|\phi\|_\ell < \infty$:

$$\phi(x+w) = \sum_{k=0}^{\ell} \frac{w^k}{k!} \phi^{(k)}(x)$$

$$+ \frac{w^{\ell+1}}{\ell!} \int_0^1 (1-s)^\ell \phi^{(\ell+1)}(x + s w)\, ds.$$

$\Rightarrow$

$$\left\| \phi(x + t^{1/2} v) - \sum_{k=0}^{\ell} \frac{t^{k/2} v^k}{k!} \phi^{(k)}(x) \right\|$$

$$= \left\| \frac{t^{(\ell+1)/2} v^{\ell+1}}{\ell!} \int_0^1 (1-s)^\ell \phi^{(\ell+1)}(x + s t^{1/2} v)\, ds \right\|$$

$$\leq \frac{t^{(\ell+1)/2}}{\ell!} |v|^{\ell+1} \|\phi\|_{\ell+1}$$

$\Rightarrow$

$$\left\| Q_t \phi - \sum_{k=0}^{l} a(k) \frac{t^{k/2}}{k!} \phi^{(k)}(x) \right\| \leq O\left(t^{(l+1)/2}\right).$$

But for k even:

$$a(k) \frac{t^{k/2}}{k!} \phi^{(k)}(x) = \frac{(-t)^k}{k!} \Delta^{k/2} \phi(x).$$

$\boxtimes$

More generally, if V is any Euclidean vector space, there is an asymptotic expansion in powers of $t^{1/2}$:

$$\int_V e^{-\|v\|^2/4t} \phi(v)\, dv \sim (4\pi t)^{\dim V/2} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} (\Delta^k \phi)(0).$$

We will show that there is an analog of the heat kernel $q_t(x,y)$ defined on any compact Riemannian manifold.