January 25

- Bundles and connections

- If E is a vector bundle of rank r we denote by $\mathcal{E}$ its space of C^∞-sections

$$\mathcal{E} = \Gamma(M, E).$$

- Let Fr_E be the $GL(r, \mathbb{R})$ principal bundle of frames. Then:

$$E \cong Fr_E \times^{GL(r,\mathbb{R})} \mathbb{R}^r.$$

- A covariant derivative on E is a linear operator

$$\nabla : \Gamma(M, E) \to \Gamma(M, T^*_M \otimes E)$$

s.t.

$$\nabla(fs) = df \otimes s + f \nabla s$$

for all $f \in C^\infty(M)$, $s \in \Gamma(M, E)$.

Let

$$\Omega^k(M, E) \overset{dfn}{=} \Gamma(M, \Lambda^k T^*_M \otimes E).$$

Thm ∇ extends to a unique operator

$$\nabla : \Omega^\bullet(M, E) \longrightarrow \Omega^{\bullet+1}(M, E)$$

s.t. $\nabla(\alpha \wedge \theta) = d\alpha \wedge \theta + (-1)^k \alpha \wedge \nabla\theta$.

- If $X \in \mathrm{Vect}(M)$, $s \in \Gamma(M, E)$ let

$$\nabla_X s = \iota_X \nabla s \in \Gamma(M, E).$$

<u>Prop</u>: If ∇, ∇' are covariant derivatives on E then for all $s \in \mathcal{E}$:

$$\nabla' s = \nabla s + \omega \cdot s$$

for some $\omega \in \Omega^1(M, \mathrm{End}\, E)$.

In particular if $E = M \times \mathbb{R}^r$ then any connection ∇ on E is of the form

$$\nabla = d + \omega$$

for some $\omega \in \Omega^1(M) \otimes \mathfrak{gl}(r)$.

- Curvature $F \in \Omega^2(M, \operatorname{End} E)$ is

$$F(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

Prop: $\nabla^2 \alpha = F \wedge \alpha$ for all $\alpha \in \Omega^\bullet(M,E)$.

$$\underset{\Omega^{2+\bullet}(M,E)}{\cap}$$

- Parallel transport. Sps $\phi : N \to M$ is smooth, let (E, ∇) be a covariant deriv. on M. Let

$$\nabla^{\phi^* E}\left(f \cdot \phi^* s\right)$$

$$= df \otimes \phi^* s + f \phi^*(\nabla s).$$

This defines a covariant derivative on the bundle $\phi^* E$ on N.

Let $\gamma : \mathbb{R} \to M$ be a smooth curve, w/ velocity $\dot{\gamma}(t) \in T_{\gamma(t)} M$, at $t \in \mathbb{R}$.

If $s : \mathbb{R} \to \gamma^* E$ is a smooth section, then let

$$\nabla_{\dot{\gamma}(t)} s(t) = \left(\nabla^{\gamma^* E} s\right)(t)$$

be the induced covariant derivative. In particular if E is trivial $E = M \times V$ and $\nabla = d + \omega$ then

$$\nabla_{\dot{\gamma}(t)} s(t)$$

$$= \left(\gamma(t), \frac{df(t)}{dt} + \omega(\dot{\gamma}(t)) f(t) \right)$$

whe $s(t) = \left(\gamma(t), f(t) \right)$.

The <u>parallel</u> <u>transport</u> along γ is

$$\tau_{\gamma(t)} \in \mathrm{Hom}\left(E_{\gamma(0)}, E_{\gamma(t)} \right)$$

the solution of the ODE

$$\nabla_{\dot{\gamma}(t)} \tau_{\gamma}(t) = 0,$$

s.t. $\tau_{\gamma(0)} = \mathbb{1}$.

Let $\mathcal{U} \subset \mathbb{R}^n$ be open ball and let

$$E_{\mathcal{U}} = \sum_i x^i \frac{\partial}{\partial x^i}$$

be the Euler vf. Parallel transport along

$$\gamma_v : \mathbb{R} \longrightarrow \mathcal{U} \quad , \quad v \in \mathcal{U}$$
$$t \longmapsto tv$$

gives trivialization of E over $\mathcal{U}$ by identifying E_v w/ E_0.

In terms of this trivialization we can write

$$\nabla = d + \omega \quad , \quad \omega \in \Omega^1(\mathcal{U}, \mathrm{End}\, E)$$

and

$$F = d\omega + \omega \wedge \omega .$$

Notice

$$L_{E_u}\omega = (i_{E_u}d + d\,i_{E_u})\omega$$

$$= i_{E_u}d\omega \quad , \quad \text{since } i_{E_u}\omega = 0$$

$$= i_{E_u}F \quad , \quad \text{since } i_{E_u}\omega = 0.$$

m)

Prop: The taylor expansion of $\omega = \sum_i \omega_i dx^i$ is of the form

$$\omega_i(x) \sim \frac{1}{2}\sum_j F(\partial_i, \partial_j)_{x_0} x^j$$

$$+ \sum_{|\alpha| \geq 2} \partial^\alpha \omega_i(x_0) \frac{x^\alpha}{\alpha!}$$

Pf: Taylor expand $L_{Eu}\omega$.

$$L_{Eu}\omega = L_{Eu}\left(\sum_{\alpha,l} \frac{1}{\alpha!}\partial^\alpha \omega_l(x_0)\, x^\alpha\right) dx^l.$$

α is multi-index.

$$= \sum_\alpha \frac{1}{\alpha!}\left(1+|\alpha|\right)\partial^\alpha \omega_l(x_0)\, x^\alpha\, dx^l.$$

Taylor expand $\iota_{Eu}F$:

$\iota_{Eu}F =$

$$\iota_{Eu}\left(\sum_{\alpha,k,l}\frac{1}{\alpha!}\partial^\alpha F(\partial_k,\partial_l)_{x_0} x^\alpha\, dx^k dx^l\right).$$

$$= \sum_{\alpha,k,l}\frac{1}{\alpha!}\partial^\alpha F(\partial_k,\partial_l)_{x_0} x^k x^\alpha\, dx^l$$

Equating dx^l components $\Rightarrow$

$$\sum_{\alpha} (1+|\alpha|)\, \partial^{\alpha} \omega_{l}(x_0) \frac{x^{\alpha}}{\alpha!}$$

$$= \sum_{\alpha, k} \partial^{\alpha} F(\partial_k, \partial_l)_{x_0} x^k \frac{x^{\alpha}}{\alpha!}$$

Equating coefficients of x^{α} we find that

$$\partial_j \omega_i (x_0) = \frac{1}{2} F(\partial_j, \partial_i)_{x_0}$$

$$= -\frac{1}{2} F(\partial_i, \partial_j)_{x_0}.$$

$\blacksquare$

• Overview of Riemannian geometry.

A Riemann structure is a metric on the tangent bundle TM.

⇝ Reduction of structure of frame bundle to $O(n) \subset GL(n)$:

$$TM = Fr_{O(M)} \overset{O(n)}{\times} \mathbb{R}^n .$$

and if M is oriented

$$TM = Fr_{SO(M)} \overset{SO(n)}{\times} \mathbb{R}^n . \qquad //$$

Levi-Civita connection:

Thm: A Riemannian mfld has a unique canonical connection ∇ on TM s.t.

1) $d(X,Y) = (\nabla X, Y) + (X, \nabla Y)$

2) Torsion-free $\Leftrightarrow$ $[X,Y] = \nabla_X Y - \nabla_Y X$.

Pf: Defined by

$$2(\nabla_X Y, Z) = ([X,Y], Z) + \cdots \text{perm} \cdots$$

$$+ X(Y,Z) + \cdots \text{perm} \cdots \quad \boxtimes$$

- Let $so(M)$ be the bundle of Lie alg's:

$$so(M) = Fr_{SO(M)} \times^{SO(n)} so(n).$$

R = curvature of LC connection

$$\in \Omega^2(M, so(M)).$$

Has many symmetries since L.C. connection is so rigid. For X, Y v.f's write

$$R(X,Y) \in \Gamma(M, \mathfrak{so}(M)).$$

① $R(X,Y) = -R(Y,X)$.

② $\big(R(X,Y)Y, Z\big) + \big(Y, R(W,X)Z\big) = 0$.

③ $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

④ $\big(R(W,X)Y, Z\big) = \big(R(Y,Z)W, X\big)$.

See Proposition 1.26 from textbook.

If: $\{X_i\}$ is $\overset{\text{local}}{\vee}$ frame for TM

$\{x^i\}$ dual $\overset{\text{local}}{\vee}$ frame for T^*M.

$\rightsquigarrow$ R is locally determined by

$$R^i_{jkl} = \langle R(X_k, X_l) X_j, x^i \rangle .$$

Using the metric:

$$R_{ijkl} = \big(R(X_k, X_l) X_j , X_i \big) .$$

- Ricci tensor:

$$
\begin{tikzcd}
R \arrow[rr, mapsto] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & Ric \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
\Omega^2(M, so(M)) \arrow[rr] \arrow[d, phantom, "\cap"] & & \Gamma(M, S^2(T^*M)) \arrow[d, phantom, "\cap"] \\
\Omega^2(M, \operatorname{End} TM) \arrow[rr] \arrow[d, phantom, "\parallel"] & & \Gamma(M, T^*M \otimes T^*M) . \\
\Gamma(M, \Lambda^2 \overbrace{T^*M \otimes TM} \otimes T^*M) & &
\end{tikzcd}
$$

In components

$$\mathrm{Ric}_{ij} = R^k_{\ ikj} \,.$$

• <u>Scalar curvature</u>

$$r = \mathrm{Ric}_{ii} = R^k_{\ iki} \,.$$

which is just a function on M.

• <u>Gradient</u>: $f \in C^\infty(M)$ define

$$\mathrm{grad}\, f \in \mathrm{Vect}(M)$$

by $$(\mathrm{grad}\, f, X) = X \cdot f \,.$$

• <u>Exponential</u>: a smooth path

$$x_t : [0,1] \longrightarrow M$$

is a <u>geodesic</u> if $\nabla_{\dot{x}_t} \dot{x}_t = 0$.

Note this is a second-order differential equation. So, it has a unique soln for small t provided we supply an initial conditions $x_{t=0} = x_0$, $\underline{x} = \dot{x}_{t=0} \in T_{x_0}M$.

The **exponential** of $\underline{x}$ at x_0 is

$$x_1 = \exp_{x_0}(\underline{x}).$$

Well-defined for $\underline{x}$ small enough.

Note $d\exp_{x_0}\big|_{x_0} = \mathbb{1} \Rightarrow$ by the implicit fn theorem $\exp_{x_0}$ is a diffeomorphism

small ball around 0

$$\begin{array}{ccc} B & \xrightarrow{\cong} & U \\ \cap & & \cap \\ T_{x_0}M & & M \end{array}$$

some nbd of x_0.

We call the induced coordinates on $U \subset M$ the <u>normal coordinate system</u>

<u>Properties</u>: ① In this coordinate, radial paths

$$t \longmapsto (tv^1, \dots, tv^n)$$

are <u>exactly</u> the geodesics.

② The metric at x_0 is

$$g(x_0) = \mathbb{1} \iff g_{ij}(x_0) = \delta_{ij}.$$

in this coordinate system.

③ In fact, for $\underline{x} \in B \subset T_{x_0}M$:

$$g_{ij}(\underline{x}) = \delta_{ij} - \frac{1}{3} \sum_{k,l} R_{ikjl}(x_0)\, \underline{x}^k \underline{x}^l + O(\underline{x}^3).$$