January 28 : We have defined the homomorphism :

$$\widetilde{Ad} : \tilde{P}(V, q) \longrightarrow O(V, q) . \quad (*)$$

Thm : [Cartan-Dieudonné] Any orthogonal transformation of V can be expressed as

$$R_{v_1} \cdots R_{v_l} \; , \; v_i \in V$$

$$l \leq n = \dim V .$$

where R_v = reflection about $v^\perp$.

In particular, this implies that $(*)$

$$P(V, q) \hookrightarrow \tilde{P}(V, q) \xrightarrow{\widetilde{Ad}} O(V, q)$$

is surjective.

Similarly, if we set

$$SP(V,q) = P(V,q) \cap C\ell^+$$

then

$$\widetilde{Ad} : SP(V,q) \longrightarrow SO(V,q)$$

is also surjective.

[Indeed, $\det R_v = -1$, for all $v \in V$.
So $\det(R_{v_1} \cdots R_{v_\ell}) = 1 \iff$
ℓ is even.].

- We want to show that when $\widetilde{Ad}$ is further restricted to $Pin(V,q)$ (resp. $Spin(V,q)$) that it still surjects onto $O(V,q)$ (resp. $SO(V,q)$).

This follows from the fact that in

$$R_{v_1} \cdots R_{v_\ell} \in O(V, q)$$

we can always normalize $v_1, \ldots, v_\ell$ to have $q(v_i) = 1$ (This assumes we can solve $t^2 = 1$, for $k = \mathbb{R}, \mathbb{C}$ we are good...)

<u>Thm</u>: Let (V, q) be a f.d. v.s. w/ a non-deg. quadratic form defnd over $k = \mathbb{R}$. Then there are SES's:

$$1 \to \mathbb{Z}/2 \to \mathrm{Pin}(V, q) \xrightarrow{\widetilde{Ad}} O(V, q) \to 1$$

$$1 \to \mathbb{Z}/2 \to \mathrm{Spin}(V, q) \xrightarrow{\widetilde{Ad}} SO(V, q) \to 1.$$

Pf: Need to compute $\ker \widetilde{Ad}$.

If $v_1 \cdots v_\ell \in Pin(V, q)$ is in the kernel then $v_1 \cdots v_\ell \in k^\times \Rightarrow$

$$(v_1 \cdots v_\ell)^2 = N(v_1 \cdots v_\ell)$$
$$= N(v_1) \cdots N(v_\ell) = \pm 1.$$

Same argument for Spin. ▨

- Now, let's specialize:

$$Spin(n) = Spin(\mathbb{R}^n, \underbrace{\sum_i x_i^2}_{\|x\|^2}).$$

Thm: For $n \geq 3$:

$$1 \to \mathbb{Z}/2 \to Spin(n) \to SO(n) \to 1.$$

exhibits $Spin(n)$ as universal cover of $SO(n)$.

Steps: ① Compute $\pi_1(SO(n))$.

② Base case $n=3$. [Where we will directly show $SU(2) \simeq Spin(3)$]

③ Show $\pi_1(Spin(n)) = 0$.

Prop:

$$\pi_1(SO(n)) \cong \mathbb{Z}/2 , \quad n \geq 3 .$$

Pf: If $H \subset G$ is Lie subgrp then

$$\begin{tikzcd} H \arrow[r, hook] & G \arrow[d] \\ & G/H \end{tikzcd}$$ is an H-principal bundle.

Consider

$$H = SO(n) = \begin{pmatrix} \boxed{SO(n)} & 0 \\ 0 & 1 \end{pmatrix} \subset \underset{\substack{\| \\ G}}{SO(n+1)} .$$

Note that

$$SO(n+1) \circlearrowleft S^n = \left\{ \|x\|^2 = 1 \right\} \subset \mathbb{R}^{n+1}$$

Take $e_{n+1} = (0, \dots, 1) \in S^n$.

<u>Claim</u>: $\mathrm{Stab}(e_{n+1}) = SO(n-1)$.

[Certainly if $A \in SO(n-1)$ then $A \cdot e_{n+1} = e_{n+1}$

Conversely: if $A \in SO(n)$ and

$$e_{n+1} = A \cdot e_{n+1} = A_{1,(n+1)} e_1 + \dots + A_{n+1,n+1} e_{n+1}$$

$$\Rightarrow A_{1,n+1} = \dots = A_{n,n+1} = 0, \; A_{n+1,n+1} = 1.$$

So A is of the form

$$A = \begin{pmatrix} \boxed{} & \begin{matrix} * \\ * \\ * \end{matrix} \\ 0 \cdots 0 & 1 \end{pmatrix}$$

But $AA^T = \mathbb{1} \Rightarrow$

$$(A_{n+1,1})^2 + \cdots + (A_{n+1,n})^2 + 1^2 = 1$$

$$\Rightarrow \quad A_{n+1,1} = \cdots = A_{n+1,n} = 0. \qquad]$$

Now, if $G \circlearrowright X$ transitive (which we have) and set for $x \in X$:

$$H_x = \{ g \in G \mid g \cdot x = x \}$$

Then $X \simeq G/H_x$. So, for us

$$\begin{array}{ccc} SO(n) & \longrightarrow & SO(n+1) \\ & & \downarrow \\ & & S^n \simeq SO(n+1)/SO(n). \end{array}$$

principal $SO(n)$-bundle.

Finally, for any fiber bundle we have LES in $\pi_\bullet(-)$. So:

$$\begin{tikzcd} \cdots \arrow[r] & \pi_2(S^n) \arrow[r] \arrow[d, equal] & \pi_1(SO(n)) \arrow[r] & \pi_1(SO(n+1)) \arrow[d] \\ & 0 & & 0 = \pi_1(S^n) \end{tikzcd}$$

$n > 2$ for $n > 1$.

$\Rightarrow$ for $n > 2$ $\quad \pi_1(SO(n)) = \pi_1(SO(n+1))$.

So, we will be done if we can show that the base case:

$$\pi_1(SO(3)) \simeq \mathbb{Z}/2.$$

To prove this, we will actually show that $\mathrm{Spin}(3) \simeq SU(2)$.

By the spectral theorem, since

$A \in SO(n)$, $\|Ax\| = \|x\|$

$\Rightarrow$ eigenvalues of $A \in \{\lambda \in \mathbb{C} / \lambda\bar{\lambda} = 1\}$.

If $v \in \mathbb{R}^n$ is eigenvector for $A \in SO(n)$

then so is $\bar{v}$:

1) λ real $\Rightarrow \lambda = \pm 1$ and v is real.

2) λ cplx $\Rightarrow v + \bar{v}$, $\frac{v - \bar{v}}{i}$ are real

and A acts by rotation by $\arg(\lambda)$

in this plane.

Since $\det A = 1$, $\lambda = -1$ must occur

an even #.

For $SO(3)$ any rotation has unique eigen vector w/ eigenvalue 1. Explicitly, every element in $SO(3)$ is conjugate to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

- Let's look at the $\mathbb{C}$-vector space $\mathbb{C}^n$ w/ its Hermitian product

$$\langle v | w \rangle = \sum_i v_i \bar{w}_i .$$

$$U(n) = \{ A \mid \langle Av | Aw \rangle = \langle v | w \rangle \}$$

$$\begin{array}{ccc} U(n) & & \\ \cap & & \cup \ \det A = 1. \\ GL(n, \mathbb{C}) & & SU(n) \end{array}$$

• Let's look at $SU(2)$.

$$SU(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \;\middle|\; a\bar{a} + b\bar{b} = 1 \right\}$$

$$\cong S^3 \subset \mathbb{C}^2 = \mathbb{R}^4.$$

$$su(2) = \mathrm{Lie}(SU(2)) = \left\{ X \in gl(2,\mathbb{C}) \;\middle|\; \begin{matrix} X^\dagger = -X \\ \mathrm{tr}\, X = 0 \end{matrix} \right\}$$

Consider the inner product

$$\langle X | Y \rangle = \frac{1}{2} \mathrm{Tr}(X Y^\dagger).$$

There is the following orthonormal basis:

$$E_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$[E_i, E_j] = 2\varepsilon_{ijk} E_k.$$

Have

$$(\mathbb{R}^3, \times) \cong (su(2), [-,-])$$

Define

$$\rho : SU(2) \to \mathrm{End}(su(2))$$

$$A \longmapsto (X \mapsto \underbrace{AXA^{-1}}_{\rho_A(X)}).$$

Then

$$\langle \rho_A(X) | \rho_A(X) \rangle$$

$$= \frac{1}{2}\mathrm{Tr}(AXA^{-1}(AXA^{-1})^{\dagger})$$

$$= \frac{1}{2}\mathrm{Tr}(AXA^{-1}(A^{\dagger})^{-1}X^{\dagger}A^{\dagger})$$

$$= \frac{1}{2}\mathrm{Tr}(AXX^{\dagger}A^{-1}) = \frac{1}{2}\mathrm{Tr}(XX^{\dagger}).$$

$\Rightarrow$ We get homomorphism

$$\begin{tikzcd} \rho : SU(2) \arrow[r] \arrow[dr, dashed] & O(3) \\ & SO(3) \arrow[u, hook] \end{tikzcd}$$

Since $SU(2)$ is connected

This defines

$$\rho : SU(2) \longrightarrow SO(3)$$

What is $\ker \rho$? If $A \in SU(2)$ is s.t. $AE_j = E_j A$ for $j = 1, 2, 3$

$\underline{i=1}$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} ib & ia \\ id & ic \end{pmatrix}$

$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ic & id \\ ia & ib \end{pmatrix}$ $\Rightarrow$ $a = d$, $b = c$.

$\underline{i=2}$: $\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -b & a \\ -a & b \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} b & a \\ -a & -b \end{pmatrix}$ $\Rightarrow b = 0$

$\underline{\text{Now}}$: $\det A = 1 \Rightarrow a^2 = 1$

So $\mathbb{Z}/2 = \left\{\begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix}\right\} = \ker \rho.$

Thus, we have seen that

$$1 \longrightarrow \mathbb{Z}/2 \to \mathrm{Spin}(3) \xrightarrow{\rho} SO(3) \to 1$$

is universal cover.

We also now know $\pi_1(SO(n)) = \mathbb{Z}/2$. for $n \geq 3$. So, to finish we need to see that $\pi_1(\mathrm{Spin}(n)) = 0$, $n \geq 3$.

We have exact sequence $(n \geq 3)$:

$$1 \longrightarrow \mathbb{Z}/2 \to \mathrm{Spin}(n) \to SO(n) \to 1$$

$\Rightarrow$ $\pi_1(\mathrm{Spin}(n))$ is an index 2 subgroup of $\pi_1(SO(n)) = \mathbb{Z}/2$

$$\Rightarrow \pi_1(\mathrm{Spin}(n)) = 0.$$

This proves the theorem.