March 18

- Where we are: We have defined the concept of a spin structure on a vector bundle $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$.

A few perspectives:

1) An orientation on E and a reduction of str. from $SO(r)$ ($r = \text{rank}(E)$) to $Spin(r)$ of Fr_E^{SO}.

2) From the fibration $\rightsquigarrow$ $SO(r) \xrightarrow{i} Fr_E^{SO} \xrightarrow{\pi} M$

there is a les:

$$0 \to H^1(M; \mathbb{Z}/2) \xrightarrow{\pi^*} H^1(Fr_E^{SO}; \mathbb{Z}/2) \xrightarrow{i^*} H^1(SO(r); \mathbb{Z}/2) \xrightarrow{\delta_E} H^2(M; \mathbb{Z}/2)$$

Thm: E admits spin str. $\Leftrightarrow$

$$w_2(E) \overset{dfn}{=} \delta_E(1)$$

3) If E is a $\mathbb{C}$-vector bundle then we have a reduction of Fr_E^{SO} to Fr_E^{U}, a $U(r)$-bundle.

$$\begin{array}{ccc} \widetilde{U}(r) & \longrightarrow & Spin(2r) \\ {\scriptstyle 2:1}\downarrow & pb & \downarrow{\scriptstyle 2:1} \\ U(r) & \longrightarrow & SO(2r) \end{array}$$

So: a spin str on E

$\Leftrightarrow$ a reduction of str of Fr_E^{U} to a $\widetilde{U}(r)$-bundle.

<u>Thm</u>: A spin str. on $\mathbb{C}$-v.b. E is a line bundle L together with:

$$L^{\otimes 2} \cong \det E.$$

Pf: The bundle $\det E$ is classified by

$$M \xrightarrow{f_E} BU(r) \xrightarrow{B\det} BU(1)$$

where f_E classifies E.

Now consider the diagram

$$
\begin{array}{ccccc}
U(1) & \longleftarrow & \tilde{U}(r) & \longrightarrow & Spin(2r) \\
{\scriptstyle z \mapsto z^2}\downarrow & pb & \downarrow & pb & \downarrow \\
U(1) & \xleftarrow{\ \det\ } & U(r) & \hookrightarrow & SO(2r)
\end{array}
$$

Both squares are pullback squares, so we can view $\tilde{U}(r)$ being defined either from spin or from $U(1)$

__Claim:__ $H^1(U(1); \mathbb{Z}/2) \xrightarrow{\cong} H^1(U(r); \mathbb{Z}/2)$.

So, a reduction of str of Fr_E^U to $\tilde{U}(r)$-bundle is equivalent to a 2:1 covering

$$Q \xrightarrow{2:1} \det Fr_E^U$$

s.t. on each fiber this map is $U(1) \xrightarrow{z \mapsto z^2} U(1)$.

But now $L = Q \times_{U(1)} \mathbb{C}$ is the desired line bundle. ∎

Ex: Consider a $\mathbb{C}$-v.b. E on $\mathbb{CP}^n$ s.t.

$$\det E = \mathcal{O}(k), \quad k = \text{odd}.$$

Then E does not admit a spin str..

- Complex geometry: A rapid overview.

Two approaches (at least):

1) A complex manifold is a nice space which admits an atlas to $\mathbb{C}^n$ for which all transition fns are holomorphic.

2) An almost complex manifold is a smooth manifold X w/ endomorphism

$$J : T_X \longrightarrow T_X \quad \text{s.t.} \quad J^2 = -\mathbb{1}$$

Given J have $T_X \otimes_{\mathbb{R}} \mathbb{C} = T_X^{1,0} \oplus T_X^{0,1}$.

Similarly

$$T^*_X \otimes_{\mathbb{R}} \mathbb{C} = T_X^{*1,0} \oplus T_X^{*0,1}.$$

An almost cplx manifold is a complex manifold

$\bar\partial^2 = 0$ ("Integrable")

where $\bar\partial$ = projection of d_{dR} onto

$\Lambda^{\bullet} T^{*0,1} X$.

[We'll return to this in detail]

→ Note that when $X = \Sigma$ is a real two-dim'l manifold then an almost cplx str on Σ is always integrable.

<u>Dfn</u>: A <u>Riemann surface</u> is a smooth 2-dim'l manifold Σ together w/ $J : T\Sigma \longrightarrow T\Sigma$ s.t. $J^2 = -\mathbb{1}$.

The complex: $\underset{0}{C^{\infty}(\Sigma)} \xrightarrow{\bar\partial} \underset{1}{\Omega^{0,1}(\Sigma)} = \Gamma(\Sigma, T^{*0,1}_X)$

is the <u>Dolbeault complex</u> of $\mathcal{O}_\Sigma$. (More generally, its defined on any cplx manifold.)

<u>Thm</u>: [Čech = Dolbeault]

$$H^{\bullet}(X, \mathcal{O}_X) \cong H^{\bullet}_{\bar\partial}(X).$$

More generally, you can consider $\Omega^{0,i}(X, E)$ where E is holomorphic v.b.

1) A holomorphic v.b on X is a cplx v.b.

$$\begin{array}{c} E \\ \downarrow \pi \\ X \end{array} \quad \text{s.t.} \quad \begin{array}{l} \text{- } E \text{ is cplx mfld.} \\ \text{- } \pi \text{ is holomorphic.} \end{array}$$

2) A hol. v.b. on X is a cplx v.b. s.t. the transition fns

$$U \cap V \longrightarrow GL(r, \mathbb{C}) \subset \mathbb{C}^{r^2}$$

are holomorphic.

3) A hol. v.b. on X is a cplx v.b. w/ a first-order diff operator

$$\bar{\partial} : \Gamma(X, \Lambda^q T^{*0,1}X \otimes E) \longrightarrow \Gamma(X, \Lambda^{q+1} T^{*0,1} X \otimes E)$$

s.t. $\bar{\partial}^2 = 0$.

• Now, sps we have

$$\begin{array}{ll} L & \text{hol. line bundle} \\ \downarrow & \\ X & = \text{cplx mfld.} \end{array}$$

Then the transition fns $\left\{ g_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \longrightarrow GL(1, \mathbb{C}) = \mathbb{C}^\times \right\}$

define $[g_{\alpha\beta}] \in H^1(\{\mathcal{U}\}; \mathcal{O}_X^\times) \cong H^1(X, \mathcal{O}_X^\times)$. ← Čech cohomology w/ coefficients in the sheaf of groups $\mathcal{O}_X^\times$.

$$\Downarrow$$

$$H^1(X, \mathcal{O}_X^\times) = \left\{ \begin{array}{l} \text{holomorphic line} \\ \text{bundles} \end{array} \right\} \Big/ \cong \; = \text{Pic}(X).$$

<u>Dfn</u>: A <u>spin str.</u> on a cplx manifold X is a spin str. on the cplx vector bundle $T_X^{1,0}$. (equivalently $T_X^{*\,1,0}$).

[This is equivalent to asking that the complex tangent bundle $T_X \otimes_{\mathbb{R}} \mathbb{C}$, which we can assume has $U(n)$ framing can be reduced to $\widetilde{U}(n)$.]