March 20: Sheaves.

A **presheaf** of abelian groups on top^l space X is an assignment

$$U \underset{op}{\subset} X \overset{\mathcal{F}}{\longmapsto} \mathcal{F}(U) \ \left(= \Gamma(U, \mathcal{F}) \right)$$

together w/ $U \hookrightarrow V$

$$\rightsquigarrow \mathcal{F}(V) \longrightarrow \mathcal{F}(U).$$

In other words, a presheaf is a functor

$$\mathcal{F} : \mathrm{Open}(X) \longrightarrow Ab.$$

A **sheaf** is a presheaf $\mathcal{F}$ s.t. for every $U \subset X$ and every open cover $\{U_i\}$ of U then the sequence:

$$0 \to \mathcal{F}(U) \xrightarrow{\{r_i\}} \prod_i \mathcal{F}(U_i) \xrightarrow{\{r_i - r_j\}} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is **exact**. So: for any collection of $s_i \in \mathcal{F}(U_i)$ s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ then there is a unique $s \in \mathcal{F}(U)$ s.t. $s|_{U_i} = s_i$.

In other words: sections which locally agree on intersections can be "glued" to a global section.

- Sheaf cohomology: Consider a short exact sequence of sheaves of abelian groups

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

on a space X. Then the global sections may not be exact. Specifically, $\mathcal{B}(X) \to \mathcal{C}(X)$ may fail to be surjective.

In other words: the global sections functor

$$\Gamma : \text{Sheaves}(X) \longrightarrow \text{Ab}$$
$$\mathcal{F} \longmapsto \mathcal{F}(X) = \Gamma(X, \mathcal{F})$$

is not an exact functor.

- Its failure to be exact is sheaf cohomology.

We will focus on a specific model for it called Čech cohomology.

Let $\mathcal{F}$ be a sheaf of abelian groups on X.

- Let $\mathcal{U} = \{U_\alpha\}$ be a cover for a top'l space X. Define

$$\check{C}^0(\mathcal{U}, \mathcal{F}) = \prod_\alpha \mathcal{F}(U_\alpha)$$

$$\check{C}^1(\mathcal{U}, \mathcal{F}) = \prod_{\alpha<\beta} \mathcal{F}(U_\alpha \cap U_\beta)$$

$$\vdots$$

$$\check{C}^q(\mathcal{U}, \mathcal{F}) = \prod_{\alpha_0<\cdots<\alpha_q} \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_q}).$$

Defn - $\delta : \check{C}^0 \longrightarrow \check{C}^1$

$$(\delta f)_{\alpha\beta} = f_\alpha - f_\beta.$$

- $\delta : \check{C}^q \longrightarrow \check{C}^{q+1}$

$$(\delta f)_{\alpha_0\cdots\alpha_{q+1}} = \sum_{i=0}^{q+1} (-1)^i f_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_{q+1}}.$$

Ex: $\delta^2 = 0$.

The cochain complex $\left(\check{C}^\bullet(\mathcal{U}, \mathcal{F}), \delta \right)$

is the Čech cochain cplx associated to $\mathcal{U}, \mathcal{F}$.

$$\check{H}^q(\mathcal{U}, \mathcal{F}) = \frac{\ker \delta|_{\check{C}^q}}{\operatorname{im} \delta|_{\check{C}^{q-1}}}.$$

- Observe that if $\mathcal{U}$ is any cover then:

$$0 \to \mathcal{F}(X) \longrightarrow \check{C}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \check{C}^1(\mathcal{U}, \mathcal{F})$$

is exact by sheaf property (this <u>is</u> the sheaf property). Thus:

$$\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X).$$

- The Čech cohomology depends on the cover chosen. To get a canonically defnd object one takes a limit over refinements.

$$\mathcal{V} = \{V_\beta\}_{\beta \in J} \text{ refines } \mathcal{U} = \{U_\alpha\}_{\alpha \in I}$$

if $\forall \beta \in J \ \exists \alpha \in I$ s.t. $V_\beta \subset U_\alpha$.

Refinements form an inverse system.

$$\check{H}^q(X, \mathcal{F}) \overset{\text{def}}{=} \operatorname*{colim}_{\mathcal{U}} \check{H}^q(\mathcal{U}, \mathcal{F}).$$

<u>Thm</u>: [In algebraic geometry] For quasi-coherent sheaves on "nice" schemes

$$\underbrace{H^q(X, \mathcal{F})}_{\text{sheaf cohomology}} \cong \underbrace{\check{H}^q(X, \mathcal{F})}_{\text{Čech cohomology.}}$$

<u>Thm</u> [Čech - de Rham] Let M be smooth manifold and $\mathcal{U}$ a "good cover" in the sense that all finite intersections are contractible. Then

$$H^q_{dR}(M) \cong \check{H}^q(\mathcal{U}, \underline{\mathbb{R}})$$

constant sheaf.

Now, since $H^1(\mathbb{P}^1; \mathbb{Z}/2) = 0$ this is the unique spin structure. $\boxtimes$

Thm: [Atiyah]

$$\begin{array}{ccc} & & 4^g \\ & & \text{SII} \\ \left\{ \begin{matrix} \text{spin structures} \\ \text{on } \Sigma_g \end{matrix} \right\} \Big/ \sim & \overset{\sim}{\longleftrightarrow} & H^1(\Sigma; \mathbb{Z}/2) \\ & \nwarrow \!\!\searrow \sim \qquad \swarrow\!\!\nearrow & \\ & \left\{ \begin{matrix} \text{"}\theta\text{-characteristics"} \\ L^{\otimes 2} \simeq K_\Sigma \end{matrix} \right\} \Big/ \sim & \end{array}$$

Pf: The bit we haven't fully justified is existence of spin str on Σ_g.

$$c_1 : \left\{ \begin{matrix} \mathbb{C}\text{-line} \\ \text{bundles} \end{matrix} \right\} \xrightarrow{\simeq} H^2(X; \mathbb{Z}).$$

But $c_1(K_\Sigma) \in 2\mathbb{Z}$, so we can always find $\mathbb{C}$-line bundle L (unique up to $\simeq$) s.t. $L^{\otimes 2} = K_\Sigma$. Note that $\boxtimes$

$\mathbb{C}$-line bundles are holomorphic on Σ.