March 27:

- **Stepping back:** we see that $H^1(\Sigma; \mathcal{O}_\Sigma^\times)$ classifies line bundles. How does this relate to Chern class?

Consider the SES of sheaves of ab groups

$$0 \to \underline{2\pi i \mathbb{Z}} \to \mathcal{O}_\Sigma \xrightarrow{\exp} \mathcal{O}_\Sigma^\times \to 0.$$

Induces LES in sheaf cohomology

$$0 \to H^1(\Sigma; 2\pi i \mathbb{Z}) \to H^1(\Sigma; \mathcal{O}_\Sigma) \to H^1(\Sigma; \mathcal{O}_\Sigma^\times) \xrightarrow{\delta} H^2(\Sigma; 2\pi i \mathbb{Z}) \to \cdots$$

The connecting map

$$\begin{array}{ccc} \delta : H^1(\Sigma; \mathcal{O}_\Sigma^\times) & \longrightarrow & H^2(\Sigma; 2\pi i \mathbb{Z}) \\ \wr\wr & & \wr\wr \\ \mathrm{Pic}(\Sigma) & \xrightarrow[c_1]{} & H^2(\Sigma; \mathbb{Z}) \end{array}$$

is the first Chern class.

The degree of L is defined when Σ is closed:

$$\deg L = \int_\Sigma c_1(L).$$

Theorem: [Riemann-Roch] $\deg K_\Sigma = 2-2g$.

Back to spin.

Prop: There is a unique spin str. on $S^2 = \mathbb{P}^1$.

Pf: We work out the clutching data for $K_{\mathbb{P}^1}$.

Usual chart $z = w^{-1}$. But then

$$dw = dz^{-1} = \frac{\partial(z^{-1})}{\partial z}\, dz = -z^{-2}\, dz.$$

So, the transition fn is $-z^{-2}$, or changing coordinates slightly, z^{-2}. This has a square-root z^{-1}, and we know this defines a line bundle.
(It is called $\mathcal{O}(1)$.)

Now, since $H^1(\mathbb{P}^1; \mathbb{Z}/2) = 0$ this is the unique spin structure. $\blacksquare$

Thm: [Atiyah]

$$\begin{tikzcd} \left\{ \begin{matrix} \text{spin structures} \\ \text{on } \Sigma_g \end{matrix} \right\} \Big/ \sim \arrow[r, leftrightarrow, "\sim"] \arrow[dr, leftrightarrow, "\sim"'] & H^1(\Sigma; \mathbb{Z}/2) \cong 4^g \arrow[d, leftrightarrow] \\ & \left\{ \begin{matrix} \text{"}\theta\text{-characteristics"} \\ L^{\otimes 2} \simeq K_\Sigma \end{matrix} \right\} \Big/ \sim \end{tikzcd}$$

Pf: The bit we haven't fully justified is existence of spin str on Σ_g.

$$c_1 : \left\{ \begin{matrix} \mathbb{C}\text{-line} \\ \text{bundles} \end{matrix} \right\} \xrightarrow{\simeq} H^2(X; \mathbb{Z}).$$

But $c_1(K_\Sigma) \in 2\mathbb{Z}$, so we can always find $\mathbb{C}$-line bundle L (unique up to $\simeq$) s.t. $L^{\otimes 2} = K_\Sigma$. Note that $\blacksquare$

$\mathbb{C}$-line bundles are holomorphic on Σ.

• Why are spin structures / θ-characteristics interesting?

Riemann/Atiyah: $q : \left\{ \begin{array}{c} \text{spin sts} \\ \text{on } \Sigma \end{array} \right\} \longrightarrow \mathbb{Z}/2$

$$L \longmapsto \dim H^0(L) \bmod 2.$$

is a quadratic function and <u>refines</u> the cup product

$$H^1(\Sigma; \mathbb{Z}/2) \times H^1(\Sigma; \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2 = H^2(\Sigma; \mathbb{Z}/2).$$

So: spin structures allow us to <u>refine</u> basic top'l invariants!

- Twistor space.

Let Σ be a 2-dim^l smooth manifold.
Two Riemannian metrics g, g' are conformally equivalent if $\exists\ \lambda : \Sigma \to \mathbb{R}_{>0}$ smooth s.t.

$$g' = \lambda \cdot g.$$

$\rightsquigarrow$ $[g]$ = conformal equivalence class of metrics.

Conformal class of metric + orientation $\overset{\cong}{\longleftrightarrow}$ complex structure on Σ.

Concretely: A conformal class of metric and an orientation defines

$$\underset{\substack{\parallel \\ \text{Rotation} \\ \text{by } 90^\circ.}}{J} : T_\Sigma \longrightarrow T_\Sigma$$

Sps now that we just have a conformal class of metric $[g]$ on Σ but not necessarily an orientation.

Consider

$$\begin{tikzcd} j \in Z = \bigcup_{p \in \Sigma} \left\{ j : T_p\Sigma \circlearrowleft \;\middle|\; \begin{array}{l} j^2 = -1 \\ j^* g = g \end{array} \right\} \arrow[d, "\pi"] \\ p \in \Sigma \end{tikzcd}$$

<u>Claim:</u> $\begin{array}{c} Z \\ \pi \downarrow \\ \Sigma \end{array} \simeq \begin{array}{c} Or_\Sigma \\ \downarrow 2:1 \\ \Sigma \end{array} =$ orientation space.

(Recall, sections of $Or_\Sigma \leftrightarrow$ orientations of Σ).

<u>So:</u> A section of $\begin{array}{c} Z \\ \downarrow \\ \Sigma \end{array}$ determines a complex structure on Σ!

We will see a higher dimensional generalization!

· <u>Almost cplx str.</u> : Two special features in 2-dim :

① An almost cplx str. on Σ^2 is automatically "integrable".

② Conf. classes of metric + orientation determines a cplx str.

Neither is true in general. Let M^{2n} be smooth.

· Given an almost cplx str. $J : T_M \circlearrowleft$, $J^2 = -1$.

define

$$\left.\begin{array}{l} T_M \otimes \mathbb{C} = \overset{+i}{T^{1,0}_M} \oplus \overset{-i}{T^{0,1}_M} \\ \\ T^*_M \quad '' \qquad '' \qquad\qquad '' \qquad\qquad '' \end{array}\right\} \overline{T^{1,0}_M} = T^{0,1}_M .$$

$$\begin{tikzcd}
\Gamma(M, \Lambda^q T^{*0,1}_M) \arrow[r, dashed, "\bar\partial"] \arrow[d, hook] & \Gamma(M, \Lambda^{q+1} T^{*0,1}_M) \\
\Gamma(M, \Lambda^q T^*_M \otimes \mathbb{C}) \arrow[r, "d_{dR}"'] & \Gamma(M, \Lambda^{q+1} T^*_M \otimes \mathbb{C}) \arrow[u, two heads]
\end{tikzcd}$$

In local coordinates:

$$\bar{\partial} = \sum_i d\bar{z}_i \frac{\partial}{\partial \bar{z}_i}.$$

Almost cplx str.

Thm: [Newlander - Nirenberg]. TFAE about (M, J):

1) M admits a cplx str

2) J is integrable: we can find atlas s.t. locally J is of the form

$$J\Big|_U = \sum_j \left(\frac{\partial}{\partial y_j} \otimes dx_j - \frac{\partial}{\partial x_j} \otimes dy_j \right).$$

$z_j = x_j + i y_j$

3) $[T^{0,1}_M, T^{0,1}_M] \subset T^{0,1}_M$

4) $\bar{\partial}^2 = 0$.

Ex: • S^{2n} admits an AC str. $\iff$ $n = 1, 3$.

• S^6 has an AC str. coming from octonions.

(This AC is not integrable.)

· Sps $V \cong \mathbb{R}^{2n}$ is an oriented v.s.. An almost cplx str $J : V \to V$ is compatible w/ this orientation if $\exists$ an oriented basis of the form

$$\{e_1, Je_1, e_2, Je_2, \dots, e_n, Je_n\}.$$

cptble w/ std or.

Ex: $V = \mathbb{R}^4$. $J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ Yes!

$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ No!

(Any metric cptble almost cplx str. of $\mathbb{R}^4$ is represented by one of these J's.)

· Some Riem geometry in 4-dimensions.

(M^4, g) oriented Riem. 4-manifold.

$\rightsquigarrow$ $* : \Omega^2(M) \longrightarrow \Omega^2(M)$, $*^2 = 1$.

So, $\Lambda^2 T^*_M = \underset{+1}{\Lambda^2_+ T^*_M} \oplus \underset{-1}{\Lambda^2_- T^*_M}$

$$\Omega^2_\pm (M) = \Gamma(M, \Lambda^2_\pm T_M).$$

Let's use g to give $T_M \underset{g}{\cong} T^*_M$. Then, locally

$$J_+ = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix} \longleftrightarrow \overbrace{dx \wedge dy + dz \wedge dw}^{=\,\omega_+}.$$

$$J_- = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix} \longleftrightarrow \overbrace{dx \wedge dy - dz \wedge dw}^{=\,\omega_-}.$$

Have $\ast\omega_\pm = \pm\omega_\pm$.

- 4d "twistor" space: Let $(M, [g])$ be oriented, conformal 4-manifold

$$\begin{tikzcd} Z = \bigcup_{p\in M} \left\{ j : T_pM \circlearrowleft \;\middle|\; \begin{array}{l} \cdot\ j^2 = -\mathbb{1} \\ \cdot\ j^*g = g \\ \cdot\ j \text{ orientation cptble.} \end{array} \right\}. \arrow[d, "\pi"'] \\ M \end{tikzcd}$$

<u>Fibers of π?</u> We have seen that

$$\Rightarrow \quad \begin{tikzcd} Z \arrow[d, "\pi"'] \\ M \end{tikzcd} \cong \begin{tikzcd} \mathrm{Sph}(\Lambda^2_+) \arrow[d] \\ M \end{tikzcd}$$

In particular π is an S^2-fibration over M.