March 4.

- Orientations: An orientation on a vector space V is an equivalence class of basis

$$[e_1, \dots, e_n] \quad \text{where}$$

$$[e_1, \dots, e_n] = [f_1, \dots, f_n] \iff \vec{f} = A\vec{e}, \ \det A > 0.$$

An orientation on a vector bundle $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ is an orientation of E_x, $x \in M$ s.t. there exists local triv's

$$E\big|_{U} \xrightarrow{\cong} U \times \mathbb{R}^n$$

which relate these orientations to the standard orientation of $\mathbb{R}^n$.

Suppose E is a Riemannian vector bundle.
$\rightsquigarrow$ Fr^O_E is its principal $O(n)$-bundle of orthonormal frames.

$$SO(n) \circlearrowright Fr^O_E$$

We form the quotient bundle:

$$\begin{array}{ccc} Or_E \overset{dfn}{=} Fr_E^O / SO(n) & & \mathbb{Z}/2 = O(n)/SO(n) \text{ - principal bundle} \\ \downarrow 2:1 & & \Updownarrow \\ M & & 2:1 \text{ covering space.} \end{array}$$

Let $w_1(E) = [Or(E)] \in H^1(M; \mathbb{Z}/2)$.

<u>Prop</u>: E is orientable $\Leftrightarrow w_1(E) = 0$.

<u>Pf</u>: Sps we have an orientation.

$$x \in M \longmapsto [e_1|_x, \dots, e_n|_x]$$

determines a section of Or_E since $SO(n)$ are those orth. matrices w/ $\det > 0$.

$\Leftrightarrow$ triv. of $Or_E \Leftrightarrow w_1(E) = 0$. ▨

· Another way to phrase orientability: E Riem

$\Rightarrow$ classified by $f_E : M \longrightarrow BO(n)$

Prop: E is orientable $\iff$ $\exists$ lift

$$\begin{tikzcd} & BSO(n) \arrow[d] \\ M \arrow[ur, dashed] \arrow[r, "f_E"'] & BO(n) \end{tikzcd}$$

This is general. A reduction of str. of $\begin{matrix} P \\ \downarrow G \\ M \end{matrix}$

along a homomorphism $\rho : H \longrightarrow G$ is:

1) A principal H-bundle $\begin{matrix} \tilde{P} \\ \downarrow H \\ M \end{matrix}$

2) An isomorphism $\tilde{P} \times_H G \cong P$.

$$\tilde{P} \times_H G = \left\{ (\tilde{p}, g) \;\middle|\; \begin{matrix} (\tilde{p} h^{-1}, \rho(h) g) \\ \shortparallel \\ (\tilde{p}, g) \end{matrix} \right\}.$$

Prop: Sps H, G are compact.

$$\left\{ \begin{tikzcd} & BH \arrow[d, "Bj"] \\ M \arrow[ur, dashed, "\tilde{f}_P"] \arrow[r, "f_P"'] & BG \end{tikzcd} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Reduction of str.} \\ \text{of } P \text{ along} \\ j : H \to G \end{matrix} \right\}.$$

Note that given $\rho : H \to G$ we have pull-back square.

$$\begin{tikzcd} EH \times_\rho G \arrow[r] \arrow[d] & EG \arrow[d] \\ BH \arrow[r, "B\rho"'] & BG \end{tikzcd}$$

So: $EH \times_\rho G \simeq (B\rho)^*(EG)$

Now, given lift $\tilde{f}$ and look at $\tilde{P} = \tilde{f}^* EH$.

Then by naturality:

$$\tilde{P} \times_\rho G = \tilde{f}^* EH \times_\rho G = \tilde{f}^*(EH \times_\rho G)$$

$$= (B\rho \circ \tilde{f})^* EG = f^* EG = P.$$

Sps $\tilde{P}$ is a ρ-reduction. Then

$$\begin{tikzcd} \tilde{P} \arrow[r] \arrow[d] & EH \arrow[d] \\ M \arrow[r, "\tilde{f}"'] & BH \end{tikzcd}$$

$$\begin{tikzcd} & & (B\rho)^* EG \arrow[d, phantom, "\wr\wr"] & \\ \rightsquigarrow & \tilde{P} \times_\rho G \arrow[r] \arrow[d] & EH \times_\rho G \arrow[r] \arrow[d] & EG \arrow[d] \\ & M \arrow[r, "\tilde{f}"'] & BH \arrow[r, "B\rho"'] & BG \end{tikzcd}$$

$\square$

So $f \simeq B\rho \circ \tilde{f}$.

Ex: $H = SO(n) \hookrightarrow O(n) = G$. Then:

$$P \times_{O(n)} \left(O(n)/SO(n) \right) \cong Or_E$$

So:

$w_1(E) = 0 \iff \exists$ lift

$$\begin{tikzcd} & BSO(n) \arrow[d] \\ M \arrow[r, "Fr_E^O"'] \arrow[ur, dashed, "Fr_E^{SO}"] & BO(n) \end{tikzcd}$$

$\Updownarrow$

E orientable

- $w_1(E)$ is an example of a characteristic class.

$$\begin{tikzcd} w_1(E) & w_1 = w_1\left(EO(n)/SO(n) \right) \arrow[l, mapsto, "f_E^*"'] \\ H^1(M; \mathbb{Z}/2) \arrow[u, phantom, "\in" {rotate=90}] & H^1(BO(n); \mathbb{Z}/2) \arrow[l] \arrow[u, phantom, "\in" {rotate=90}] \end{tikzcd}$$

universal class.

$$f_E : M \longrightarrow BO(n)$$

It turns out:

$$|w_i| = i.$$

$$H^\cdot(BO(n); \mathbb{Z}/2) \simeq \mathbb{Z}/2[w_1, w_2, \dots, w_n]$$

- <u>Spin structures</u>:

Dfn: A <u>spin str.</u> on an oriented (Riem.) vector bundle E is a reduction of str. of Fr_E^{SO} along $p: Spin(n) \longrightarrow SO(n)$.

In other words, this is principal $Spin(n)$ bundle $\tilde{P}$ on M s.t.:

$$\begin{tikzcd} \tilde{P} \arrow[rr, "2:1"] \arrow[dr] & & Fr_E \arrow[dl] \\ & M & \end{tikzcd}$$

Spin(n)-equivariant.

Equivalent if:

$$\begin{tikzcd} \tilde{P} \arrow[rr, "\simeq"] \arrow[dr] \arrow[ddr, bend right] & & \tilde{P}' \arrow[dl] \arrow[ddl, bend left] \\ & Fr_E^{SO} \arrow[d] & \\ & M & \end{tikzcd}$$

This is equivalent to the choice of a lift (up to homotopy)

$$\begin{tikzcd} & BSpin(n) \arrow[d] \\ M \arrow[ur, dashed] \arrow[r, "f_E"'] & BSO(n) \end{tikzcd}$$

- Relation w/ $w_2(E)$ (what is $w_2(E)$?).

Recall $w_1 \in H^1(BO(n); \mathbb{Z}/2)$.

$$\begin{array}{c} H^1(BO(n); \mathbb{Z}/2) \\ \| \wr \\ [BO(n), K(\mathbb{Z}/2, 1)] \end{array}$$

This is witnessed by

$$\begin{tikzcd} BSO(n) \arrow[r] & BO(n) \arrow[r, "w_1"] & B\mathbb{Z}/2 = K(\mathbb{Z}/2,1) \\ & M \arrow[u, "f_E"'] \arrow[ul, dashed] & \end{tikzcd}$$

$$w_1 \circ f_E \simeq *$$
$$\Updownarrow$$
$$w_1(E) = 0.$$

Now, w_2 is defined as follows.

$$K(\mathbb{Z}/2, 1) \longrightarrow BSpin(n) \xrightarrow{Bj} BSO(n)$$

So the cofiber of Bj is $K(\mathbb{Z}/2, 2)$:

$$\begin{tikzcd}
BSpin(n) \arrow[r] & BSO(n) \arrow[r, "w_2"] & K(\mathbb{Z}/2, 2) \\
 & M \arrow[u, "f"'] \arrow[ul, dashed] &
\end{tikzcd}$$

$w_2 \circ f \simeq *$

$\iff w_2(E) = 0$.

• How to compute w_2 / find spin structures?

Prop: Sps E is a real $2n$-dim'l v.b. which underlies a $\mathbb{C}$-vector bundle V. Then

$$w_2(E) = c_1(V) \bmod 2$$

where $c_1(V) \in H^2(M, \mathbb{Z})$ is first Chern class.