March 6 : Everything today works over $\mathbb{R}$, $\mathbb{C}$, but we'll focus on $\mathbb{C}$.

The Grassmannian of k planes in a vector space V is:

$$Gr_k(V) = \{ W \subset V \mid \dim W = k \}.$$

This is a CW complex and a smooth manifold.

Ex: $Gr_1(V) \cong \mathbb{C}P^{\dim V}$. There is an inductively defined cell structure on $\mathbb{C}P^n$:

$$\begin{tikzcd} S^1 \arrow[r, "h"] \arrow[d, hook] & * \arrow[d] \\ D^2 \arrow[r] & \mathbb{C}P^1 \end{tikzcd}$$

Hopf

$$\begin{tikzcd} S^3 \arrow[r, "h"] \arrow[d, hook] & \mathbb{C}P^1 \arrow[d] \\ D^4 \arrow[r] & \mathbb{C}P^2 \end{tikzcd}$$

...

In general, the map h is:

$$\begin{tikzcd} S^{2n+1} \subset \mathbb{C}^{n+1} - 0 \arrow[r] & \mathbb{C}P \end{tikzcd}$$

with $S^{2n+1} \xrightarrow{h} \mathbb{C}P$.

<u>Infinite Grassmannia</u>: Let $\mathcal{H}$ be a seperable Hilbert space (our $\mathbb{C}$).

$$Gr_k(\mathcal{H}) = \left\{ W \subset \mathcal{H} \;\middle|\; \dim W = k \right\}.$$

This is a "Hilbert manifold" = locally a Hilbert space, glued together by smooth charts.

Another construction:

$$\begin{tikzcd}
Gr_k(\mathbb{C}^j) \arrow[r, hook] \arrow[dr] & Gr_k(\mathbb{C}^{j+1}) \arrow[r, hook] \arrow[d] & \cdots \quad Gr_k(\mathbb{C}^\infty) \arrow[dl, "\cong"] \\
& Gr_k(\mathcal{H}) &
\end{tikzcd}$$

- We have seen how to form a vector bundle $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ of rank k to a principal $GL(k)$-bundle:

$$\begin{matrix} Fr_E \\ \downarrow \\ M \end{matrix} = \text{bundle of frames of } E.$$

On the other hand, consider a principal $GL(k)$ bundle P and define:

$$P \times_{GL(k)} \mathbb{R}^k = (P \times \mathbb{R}^k) / GL(k)$$

Action is $\quad g \cdot (p, v) = (pg^{-1}, g \cdot v)$

This is naturally a rank k v.b. in fact

$$E \cong Fr_E \times_{GL(k)} \mathbb{R}^k .$$

- More generally if $\begin{array}{c} P \\ \downarrow \\ M \end{array}$ is a prinipal G-bundle and V is a G-representation then the associated vector bundle is

just as above.

$$P \times_G V = (P \times V) / G$$

- So:

$$\left\{ \begin{array}{c} \text{rank } k \text{ vector} \\ \text{bundles over } M \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Principal} \\ GL(k)\text{-bundles} \\ \text{over } M \end{array} \right\} .$$

- **Back to Grassmannians:** For V a v.s. let:

$$\begin{tikzcd} St_k(V) \arrow[d, "GL(k)"'] \\ Gr_k(V) \end{tikzcd} \qquad St_k(V) = \left\{ \begin{matrix} \text{rank } k \text{ frames} \\ \text{in } V \end{matrix} \right\} = \left\{ b : \mathbb{C}^k \hookrightarrow V \;\middle|\; b \text{ inj.} \right\}$$

Take the subspace spanned by the frame.

In fact, this is a principal $GL(k)$-bundle.

Can do the same for V replaced by Hilbert space $\mathcal{H}$.

$$\begin{tikzcd} St_k(\mathcal{H}) \arrow[d] \\ Gr_k(\mathcal{H}) \end{tikzcd} \qquad \text{principal } GL(k)\text{-bundle.}$$

Prop: $St_k(\mathcal{H})$ is contractible.

Pf: We will prove $k=1$. Then $St_1(\mathcal{H})$ is the unit sphere in $\mathcal{H}$.

$$St_1(\mathcal{H}) = S \subset \mathcal{H}$$

We show S is contractible. Let

$$D = \{ v \in \mathcal{H} \mid |v| \le 1 \}.$$

We produce a deformation retraction $D \to S$.

* Finite dim: Sps we found a fixed point free map

$$h : D^n \longrightarrow D^n$$

Then

$$r : D^n \longrightarrow S^{n-1}$$

$$r(x) = \frac{x - h(x)}{|x - h(x)|}$$

would be a deformation retraction.

Of course, Brouwer says this is impossible.

On the other hand, we can find a fixed point free map $D \to D$. First define

$$i : \mathbb{R} \hookrightarrow D$$

$$t \longmapsto \cos\left(\frac{1}{2}(t-n)\pi\right) e_n + \sin\left(\frac{1}{2}(t-n)\pi\right) e_{n+1}.$$

<u>So</u>: $i\big|_{[n,n+1]}$ is the path in S which connects $e_n \rightsquigarrow e_{n+1}$.

This is a closed map, and is homeo onto image.

<u>So</u>: D has subspace homeomorphic to $\mathbb{R}$.

$$\mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto x+1$$

can be extended to fixed point free map $D \to D$.

Inductive

$$\begin{array}{ccc} F & \longrightarrow & St_k(\mathcal{H}) \\ & & \downarrow \\ & & St_{k-1}(\mathcal{H}) \end{array}$$

$\underline{F_b = \pi^{-1}(b)}$: Sps our frame is

$$b = \{e_1, \dots, e_k\}$$

then $\pi(b) = \{e_2, \dots, e_k\}$. The fiber $\pi^{-1}(b)$ is the set of nonzero vectors in

$$\mathrm{Span}\{e_2, \dots, e_k\}^{\perp} \setminus 0 = \tilde{\mathcal{H}} \setminus 0 .$$

S/ closed subspace so its a Hilbert space $\tilde{S} \subset \tilde{\mathcal{H}}$. But $\tilde{S} \simeq *$ again.

So fibers of π are contractible. This shows π is a homotopy equivalence. $\boxtimes$

- If $G \subset \mathcal{U}(k) \subset GL(k)$ then

$$G \circlearrowright St_k(\mathcal{H}) \text{ is still free.}$$

$$\rightsquigarrow \quad \begin{array}{c} St_k(\mathcal{H}) = EG \\ \downarrow \\ BG \end{array} \quad \text{is classifying space.}$$

- **Line bundles**: Recall that $Gr_1(\mathbb{C}^n) = \mathbb{C}P^n$ has CW cplx str where we attach an even $2j$-cell for each $j = 1, \dots, n$. This leads to:

$$H^\cdot(\mathbb{C}P^n; \mathbb{Z}) \simeq \mathbb{Z}[y]/(y^{n+1}), \quad |y| = 2.$$

Similarly $Gr_1(\mathcal{H}) \simeq \mathbb{C}P^\infty = \operatorname{colim}_n \mathbb{C}P^n$

$$\rightsquigarrow \quad H^\cdot(\mathbb{C}P^\infty; \mathbb{Z}) \simeq \mathbb{Z}[y], \quad |y| = 2.$$

Normalize: If $V \cong \mathbb{C}^2 \subset \mathcal{H}$ then

$$[\mathbb{P}(V)] \in H_2(\mathbb{P}(\mathcal{H}))$$

$$\langle [\mathbb{P}(V)], y \rangle = 1.$$

Dfn: If $\begin{matrix} L \\ \downarrow \\ M \end{matrix}$ is cplx line bundle classified by

$$f_L : M \longrightarrow \mathbb{C}P^\infty$$

then $\boxed{c_1(L) = -f_L^*(y).} \in H^2(M; \mathbb{Z}).$

Prop: If $\begin{matrix} L_1, L_2 \\ \downarrow \\ M \end{matrix}$ are two cplx line bundles then

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2).$$

Pf: We prove a universal result. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and $\begin{matrix} \xi_i \\ \downarrow \\ \mathbb{P}(\mathcal{H}_i) \end{matrix}$ the universal line bundles. Have p.b. square

$$\begin{tikzcd} \xi_1 \boxtimes \xi_2 \arrow[r] \arrow[d] & \xi \arrow[d] \\ \mathbb{P}(\mathcal{H}_1) \times \mathbb{P}(\mathcal{H}_2) \arrow[r, "f"'] & \mathbb{P}(\mathcal{H}_1 \otimes \mathcal{H}_2) \end{tikzcd}$$

where $$f(\ell_1, \ell_2) = \operatorname{span}\{v_1 \otimes v_2\}$$
where $v_i \neq 0$ lie on ℓ_i.

__Now__: Sps $\ell_i \in \mathbb{P}(\mathcal{H}_i)$ and $V_i \subset \mathcal{H}$ are 2-dim^l subspaces.

Then

$$f\left(\mathbb{P}(V_1) \times \{L_2\}\right) \subset \mathbb{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

and

$$f\left(\{L_1\} \times \mathbb{P}(V_2)\right) \subset \mathbb{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

are both proj. lines in $\mathbb{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

$$\Rightarrow \quad f^*(y) = y_1 + y_2. \qquad \blacksquare$$

<u>Cor</u>: $c_1(L^*) = -c_1(L)$.

- $\det : GL(k) \longrightarrow \mathbb{C}^\times$, defines

$$\begin{tikzcd} M \arrow[r, "f_E"'] \arrow[rr, bend left, "f_{\det E}"] & BGL(k) \arrow[r, "B\det"'] & B\mathbb{C}^\times \simeq \mathbb{C}P^\infty \end{tikzcd}$$

Explicitly $\det E = \Lambda^{top} E$.

We define $c_1(E) = c_1(\det E)$.

- <u>Back to spin structures.</u>

$$\begin{tikzcd}
\tilde{U}(n) \arrow[r, dashed] \arrow[d, dashed] & \mathrm{Spin}(2n) \arrow[d] \\
U(n) \arrow[r] & SO(2n)
\end{tikzcd}$$

Double cover of $U(n)$.

* A spin str. on a cplx v.s. is the same as a reduction of str. to $\tilde{U}(n)$.

<u>Prop</u>: Sps that a complex vector bundle $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ admits a spin str. Then $\exists\ \tilde{c}$ s.t.

$$c_1(E) = 2\tilde{c}.$$

<u>Cor</u>: $\mathbb{C}P^n$ does not admit a spin str. for n even.