Spin structures in geometry

1. A topological take on orientations

A two-sheeted covering of a topological space X is a continuous projection $p\colon \widetilde{X} \to X$ such that for any $x \in X$ there exists a neighborhood $U \subset X$ of x with $p^{-1}(U) = V_1 \sqcup V_2$ and where $p|_{V_i}\colon V_i \to U$ are homeomorphisms. Two covering spaces are equivalent if there exists a homeomorphism between them which makes the obvious triangle commute. There is a natural group isomorphism between the group of equivalence classes of two-sheeted covering spaces of X and $H^1(X; \mathbf{Z}/2)$. In general, for G a finite abelian group the space BG classifies all G-bundles, in the sense that they are in bijective correspondence with homotopy classes of maps $X \to BG$. On the other hand, BG is a Eilenberg-Maclane $K(G,1)$ space so that

$$[X, BG] \simeq [X, K(G,1)] \simeq H^1(X; G). \tag{1}$$

Let $\pi\colon E \to X$ be a real rank r vector bundle equipped with a metric (a positive definite inner product continuously defined on the fibers). Let $P_O(E)$ denote the principal $O(r)$-bundle of orthonormal frames of E. Define the orientation bundle $\mathrm{Or}(E) = P_O(E)/SO(r)$. Since $SO(n) \subset O(n)$ is an index two subgroup, we see that $\mathrm{Or}(E) \to X$ is a two-sheeted covering of X.

We define first Stiefel–Whitney class of E to be the equivalence class of this orientation bundle

$$w_1(E) \overset{\text{def}}{=} [\mathrm{Or}(E)] \in H^1(X; \mathbf{Z}/2).$$

This class $w_1(E)$ is natural in the sense that for any continuous map $f\colon Y \to X$ one has $f^*w_1(E) = w_1(f^*E)$. A bundle E is classified by a map $f_E\colon X \to BO(r)$. Let $EO(r) \to BO(r)$ denote the universal r-plane bundle and set

$$w_1 \overset{\text{def}}{=} w_1(EO(r)) \in H^1(BO(r); \mathbf{Z}/2).$$

By naturality, one has $w_1(E) = f_E^* w_1$.

The most important property of the first Stiefel–Whitney class is the following.

Proposition 1.1. *A bundle E is orientable if and only if $w_1(E) = 0$. The set of orientations is in bijection with $H^0(X; \mathbf{Z}/2)$.*

To state this in terms of maps, $w_1(E) = 0$ provides the existence of a lift

$$\begin{array}{ccc} & & BSO(r) \\ & \nearrow & \downarrow \\ X & \longrightarrow & BO(r) \end{array}$$

making the diagram commute up to homotopy.

We can state this more structurally as follows. Suppose $j\colon H \to G$ is a homomorphism of Lie groups[1] and $P \to X$ is a principal G-bundle. A *reduction of structure* of P along j is a principal H-bundle $\widetilde{P} \to X$ together with an isomorphism of principal G-bundles

$$P \simeq \widetilde{P} \times^H G. \tag{2}$$

The vanishing of $w_1(E) = 0$ is equivalent to a reduction of structure $\widetilde{P} = P_{SO}(E)$ of the principal $O(n)$ bundle of frames $P = P_O(E)$ along the natural embedding $j\colon SO(r) \to O(r)$.

The map $j = j_0\colon SO(r) \to O(r)$ has a nice property. First, note that $O(r)$ is disconnected $\pi_0(O(r)) = \mathbf{Z}/2$, yet $SO(r)$ is connected. Secondly, j induces an isomorphism $\pi_m(SO(r)) \to \pi_m(O(r))$ for all $m > 0$. In other words, $SO(r)$ is a connected version of $O(r)$. What about a more connected version of $O(r)$? The homomorphism $j_1\colon Spin(r) \to SO(r)$ has the desired properties. Indeed, $Spin(r)$ is simply connected $\pi_0(Spin(r)) = \pi_1(Spin(r)) = 0$ and for $m > 1$ the induced map $\pi_m(Spin(r)) \to \pi_m(SO(r))$ is an isomorphism.

Definition 1.2. A ***spin structure*** on an oriented vector bundle $E \to X$ is a reduction of structure of the principal $SO(r)$-bundle $P_{SO}(E)$ to a principal $Spin(r)$-bundle $P_{Spin}(E)$.

Is there a characteristic class which measures this reduction of structure? Yes!

Lemma 1.3. *The fibration*

$$SO(r) \to P_{SO}(E) \to X \tag{3}$$

induces a natural exact sequence in $\mathbf{Z}/2$ cohomology

$$0 \to H^1(X; \mathbf{Z}/2) \xrightarrow{\pi^*} H^1(P_{SO}(E); \mathbf{Z}/2) \xrightarrow{i^*} H^1(SO(r); \mathbf{Z}/2) \xrightarrow{\delta} H^2(X; \mathbf{Z}/2). \tag{4}$$

[1] Note that j does not need to be injective or surjective, any homomorphism will do.

Let $g \in H^1(SO(r); \mathbf{Z}/2) = \mathbf{Z}/2$ be the generator, and define the second Stiefel–Whitney class to be its image under the connecting homomorphism appearing in the lemma

$$w_2(E) \stackrel{\text{def}}{=} \delta(g) \in H^2(X; \mathbf{Z}/2). \tag{5}$$

Again, one can show that $w_2(E)$ is natural. Furthermore,

$$w_2 \stackrel{\text{def}}{=} w_2(ESO(r)) \in H^2(BSO(r); \mathbf{Z}/2) \simeq \mathbf{Z}/2$$

is a generator and pulls back to $w_2(E)$ along the classifying map $X \to BSO(r)$.

THEOREM 1.4. *Let E be an oriented vector bundle on X. Then E has a spin structure if and only if $w_2(E) = 0$. Furthermore, if $w_2(E) = 0$, then the set of equivalence classes of spin structures is a torsor for $H^1(X; \mathbf{Z}/2)$.*

To state this in terms of maps, $w_2(E) = 0$ provides the existence of a lift

$$\begin{array}{ccc} & & BSpin(r) \\ & \nearrow & \downarrow \\ X & \longrightarrow & BSO(r) \end{array}$$

which commutes up to homotopy.

Remark 1.5. This is about as high up as we can go in the Whitehead tower by considering just ordinary Lie groups. Indeed, if G is a simply connected Lie group then $\pi_2(G) = 0$. If, furthermore, $\pi_3(G) = 0$ then G is necessarily contractible.