Spin structures in geometry

1. A topological take on orientations

A two-sheeted covering of a topological space *X* is a continuous projection $p: \tilde{X} \to X$ such that for any $x \in X$ there exists a neighborhood $U \subset X$ of x with $p^{-1}(U) = V_1 \sqcup V_2$ and where $p|_{V_i}: V_i \to U$ are homeomorphisms. Two covering spaces are equivalent if there exists a homeomorphism between them which makes the obvious triangle commute. There is a natural group isomorphism between the group of equivalence classes of two-sheeted covering spaces of *X* and $H^1(X; \mathbb{Z}/2)$. In general, for *G* a finite abelian group the space *BG* classifies all *G*-bundles, in the sense that they are in bijective correspondence with homotopy classes of maps $X \to BG$. On the other hand, *BG* is a Eilenberg-Maclane K(G, 1) space so that

(1)
$$[X, BG] \simeq [X, K(G, 1)] \simeq H^1(X; G).$$

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Let $\pi: E \to X$ be a real rank r vector bundle equipped with a metric (a positive definite inner product continuously defined on the fibers). Let $P_O(E)$ denote the principal O(r)-bundle of orthonormal frames of E. Define the orientation bundle $Or(E) = P_O(E)/SO(r)$. Since $SO(n) \subset O(n)$ is an index two subgroup, we see that $Or(E) \to X$ is a two-sheeted covering of X.

We define first Stiefel–Whitney class of *E* to be the equivalence class of this orientation bundle

$$w_1(E) \stackrel{\text{def}}{=} [\operatorname{Or}(E)] \in H^1(X; \mathbb{Z}/2).$$

This class $w_1(E)$ is natural in the sense that for any continuous map $f: Y \to X$ one has $f^*w_1(E) = w_1(f^*E)$. A bundle *E* is classified by a map $f_E: X \to BO(r)$. Let $EO(r) \to BO(r)$ denote the universal *r*-plane bundle and set

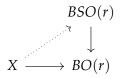
$$w_1 \stackrel{\text{der}}{=} w_1(EO(r)) \in H^1(BO(r); \mathbb{Z}/2).$$

By naturality, one has $w_1(E) = f_E^* w_1$.

The most important property of the first Stiefel–Whitney class is the following.

Proposition 1.1. A bundle *E* is orientable if and only if $w_1(E) = 0$. The set of orientations is in bijection with $H^0(X; \mathbb{Z}/2)$.

To state this in terms of maps, $w_1(E) = 0$ provides the existence of a lift



making the diagram commute up to homotopy.

We can state this more structurally as follows. Suppose $j: H \to G$ is a homomorphism of Lie groups¹ and $P \to X$ is a principal *G*-bundle. A *reduction of structure* of *P* along *j* is a principal *H*-bundle $\tilde{P} \to X$ together with an isomorphism of principal *G*-bundles

(2)

$$P\simeq \widetilde{P} imes^H G$$

The vanishing of $w_1(E) = 0$ is equivalent to a reduction of structure $\tilde{P} = P_{SO}(E)$ of the principal O(n) bundle of frames $P = P_O(E)$ along the natural embedding $j: SO(r) \to O(r)$.

The map $j = j_0: SO(r) \rightarrow O(r)$ has a nice property. First, note that O(r) is disconnected $\pi_0(O(r)) = \mathbb{Z}/2$, yet SO(r) is connected. Secondly, j induces an isomorphism $\pi_m(SO(r)) \rightarrow \pi_m(O(r))$ for all m > 0. In other words, SO(r) is a connected version of O(r). What about a more connected version of O(r)? The homomorphism $j_1: Spin(r) \rightarrow SO(r)$ has the desired properties. Indeed, Spin(r) is simply connected $\pi_0(Spin(r)) = \pi_1(Spin(r)) = 0$ and for m > 1 the induced map $\pi_m(Spin(r)) \rightarrow \pi_m(SO(r))$ is an isomorphism.

Definition 1.2. A *spin structure* on an oriented vector bundle $E \rightarrow X$ is a reduction of structure of the principal SO(r)-bundle $P_{SO}(E)$ to a principal Spin(r)-bundle $P_{Spin}(E)$.

Is there a characteristic class which measures this reduction of structure? Yes!

Lemma 1.3. The fibration

$$SO(r) \to P_{SO}(E) \to X$$

induces a natural exact sequence in $\mathbb{Z}/2$ cohomology

(4) $0 \to H^1(X; \mathbb{Z}/2) \xrightarrow{\pi^*} H^1(P_{SO}(E); \mathbb{Z}/2) \xrightarrow{i^*} H^1(SO(r); \mathbb{Z}/2) \xrightarrow{\delta} H^2(X; \mathbb{Z}/2).$

¹Note that j does not need to be injective or surjective, any homomorphism will do.

Let $g \in H^1(SO(r); \mathbb{Z}/2) = \mathbb{Z}/2$ be the generator, and define the second Stiefel– Whitney class to be its image under the connecting homomorphism appearing in the lemma

(5)
$$w_2(E) \stackrel{\text{def}}{=} \delta(g) \in H^2(X; \mathbb{Z}/2).$$

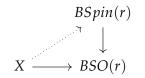
Again, one can show that $w_2(E)$ is natural. Furthermore,

$$w_2 \stackrel{\text{def}}{=} w_2(ESO(r)) \in H^2(BSO(r); \mathbb{Z}/2) \simeq \mathbb{Z}/2$$

is a generator and pulls back to $w_2(E)$ along the classifying map $X \to BSO(r)$.

THEOREM 1.4. Let *E* be an oriented vector bundle on *X*. Then *E* has a spin structure if and only if $w_2(E) = 0$. Furthermore, if $w_2(E) = 0$, then the set of equivalence classes of spin structures is a torsor for $H^1(X; \mathbb{Z}/2)$.

To state this in terms of maps, $w_2(E) = 0$ *provides the existence of a lift*



which commutes up to homotopy.

Remark 1.5. This is about as high up as we can go in the Whitehead tower by considering just ordinary Lie groups. Indeed, if *G* is a simply connected Lie group then $\pi_2(G) = 0$. If, furthermore, $\pi_3(G) = 0$ then *G* is necessarily contractible.