SPIN GEOMETRY
PROBLEM SHEET 1

Problem 1. *Filtered algebras.* Let $0 = F^{-1}A \subset F^0 A \subset \cdots \subset F^\infty A = A$ be an increasingly filtered associative algebra.

(1) Show that the algebra

$$\operatorname{gr} A \stackrel{\text{def}}{=} \oplus_{k \geq 0} F^k A / F^{k-1} A$$

is a commutative if and only if $ab - ba \in F^{p+q-1}A$ for all $a \in F^p A$, $b \in F^q A$.

(2) Suppose that $\operatorname{gr} A$ is commutative. For $a \in F^p A$, let $\overline{a} = a \mod F^{p-1}A$ be its image in $\operatorname{gr} A$. Show that the bilinear operation

$$\{-,-\} \colon \operatorname{gr} A \times \operatorname{gr} A \to \operatorname{gr} A$$

defined by

$$\{\overline{a}, \overline{b}\} \stackrel{\text{def}}{=} (ab - ba) \mod F^{p+q-2}A$$

is a Poisson bracket on $\operatorname{gr} A$. (You should first check that $\{-,-\}$ is well-defined.)

(3) State and prove a "super" version of the above result.

(4) In class, we defined an increasing filtration on $A = \mathrm{C}\ell(V, q)$–the Clifford algebra associated to the quadratic vector space (V, q). With respect to this filtration, $\operatorname{gr} \mathrm{C}\ell(V, q) = \wedge^\bullet V$. Describe the induced Poisson bracket $\{-,-\}$ in this example.

Problem 2. *Classification of complex Clifford algebras.* Let $\mathbf{C}\ell(n)$ be the Clifford algebra, defined over $\mathbf{C}$, associated to the vector space $\mathbf{C}^n$ equipped with its standard non-degenerate (holomorphic) quadratic form $q(z) = \sum_i z_i^2$.

(1) Compute $\dim_{\mathbf{C}} \mathbf{C}\ell(n)$.

(2) Show that $\mathbf{C}\ell(1) \cong \mathbf{C} \oplus \mathbf{C}$, as algebras.

(3) Consider the Pauli sigma matrices σ_i which satisfy $\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + \mathrm{i}\epsilon_{ijk}\sigma_k$. Show that $\mathrm{i}\mathbb{1}, \mathrm{i}\sigma_1, \mathrm{i}\sigma_2, \mathrm{i}\sigma_3$ form a basis for $\mathbf{C}\ell(2)$, and hence $\mathbf{C}\ell(2) \cong \operatorname{End}_{\mathbf{C}}(\mathbf{C}^2)$.

(4) Show that

$$\mathbf{C}\ell(n+2) \cong \mathbf{C}\ell(n) \otimes_{\mathbf{C}} \operatorname{End}_{\mathbf{C}}(\mathbf{C}^2).$$

(5) Let $\mathrm{C}\ell_{r,s}$ be the real Clifford algebra associated to the standard quadratic form on $\mathbf{R}^{r+s}$ of signature (r, s). Show that there is an isomorphism

$$\mathrm{C}\ell_{r,s} \otimes_{\mathbf{R}} \mathbf{C} \cong \mathbf{C}\ell(r+s).$$

Problem 3. *Changing base.*

(1) Consider the algebra $A = \mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$. Find an element $T \in A$ such that $T^2 = 1 \otimes 1$. By considering $P_{\pm} \colon A \to A$, the projections onto the ± 1 eigenspaces for the operator $T \cdot (-) \colon A \to A$, show that $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \cong \mathbf{C} \oplus \mathbf{C}$ as real associative algebras.

(2) Show that the bilinear map

$$\phi \colon \mathbf{C} \times \mathbf{H} \to \mathrm{End}_{\mathbf{C}}(\mathbf{H})$$

defined by $\phi(z, y)x = zx\overline{y}$ determines an isomorphism

$$\mathbf{C} \otimes_{\mathbf{R}} \mathbf{H} \cong \mathrm{End}_{\mathbf{C}}(\mathbf{C}^2)$$

of real associative algebras.

(3) Show that the bilinear map

$$\phi \colon \mathbf{H} \times \mathbf{H} \to \mathrm{End}_{\mathbf{R}}(\mathbf{H})$$

defined by $\phi(y_1, y_2)x = y_1 x \overline{y}_2$ determines an isomorphism

$$\mathbf{H} \otimes_{\mathbf{R}} \mathbf{H} \cong \mathrm{End}_{\mathbf{R}}(\mathbf{R}^4)$$

of real associative algebras.

Problem 4. *Pin and spin.*

(1) A *spinor representation* is the restriction of an irreducible $C\ell(V)_0$-representation to the group $Spin(V)$. Show that, as a $Spin(V)$-representation, a spinor representation is irreducible.

(2) Regard $Sp(1)$ as the group of unit quaternions $\{y \in \mathbf{H} \mid \|y\| = 1\}$, which is diffeomorphic to S^3. Describe the isomorphism

$$Spin(3) \cong Sp(1)$$

by constructing an action of $Sp(1)$ on $\operatorname{Im} \mathbf{H}$.

(3) Similarly, describe the isomorphism

$$Spin(4) \cong Sp(1) \times Sp(1)$$

by constructing an action of $Sp(1) \times Sp(1)$ on $\mathbf{H}$.

The next two parts of the problem set up the exceptional isomorphism

$$Spin(6) \cong SU(4).$$

(4) Let V be a four-dimensional complex vector space equipped with an inner product and an element $\nu \in \wedge^4 V$ such that $\|\nu\| = 1$. Show that V is equipped with a conjugate-linear isomorphism

$$\star_\nu \colon \wedge^2 V \to \wedge^2 V$$

such that $\star_\nu^2 = \mathbb{1}$ and that $\dim_{\mathbf{R}} \wedge^2_\pm V = 6$. Here, $\wedge^2_\pm V \subset \wedge^2 V$ are the ± 1 eigenspaces of $\star_\nu$.

(5) Show that the group $SU(V)$ naturally acts on $\wedge^2_\pm V$. By taking $V = \mathbf{C}^4$, argue that this leads to the isomorphism $Spin(6) \cong SU(4)$.

The next two parts of the problem set up the exceptional isomorphism

$$Spin(5) \cong Sp(2).$$

(6) Suppose now that the four-dimensional complex vector space V is equipped with a compatible complex symplectic structure $\omega \in \wedge^2 V$. Argue that one, or both, of the six-dimensional real $Sp(V)$-representations $\wedge^2_+ V, \wedge^2_- V$ contains a *five-dimensional* subspace invariant under the action of $Sp(V)$.

(7) By considering $V = \mathbf{C}^4$, and using $Spin(6) \cong SU(4)$, complete the proof that $Spin(5) \cong Sp(2)$.