Spin geometry

Problem sheet 2

Problem 1. *Ideas from Atiyah–Bott–Shapiro*

(1) Suppose that $M = M^0 \oplus M^1$ is a graded Clifford module. Show that there is an isomorphism of Clifford modules

$$M \cong \mathrm{C}\ell_n \otimes_{\mathrm{C}\ell_n^0} M^0. \tag{1}$$

(2) Let $\mathcal{M}_n^{\mathbf{C}}$ denote the Grothendieck group of graded modules for the complex Clifford algebra $\mathbf{C}\ell_n$ and let $\mathcal{A}_\bullet^{\mathbf{C}}$ be the cokernel of the homomorphism $\mathcal{M}_{n+1}^{\mathbf{C}} \xrightarrow{i^*} \mathcal{M}_n^{\mathbf{C}}$ induced from the inclusion $i\colon \mathrm{C}\ell_n^{\mathbf{C}} \to \mathrm{C}\ell_{n+1}^{\mathbf{C}}$. In class, we showed that $\mathcal{A}_\bullet^{\mathbf{C}}$ is a $\mathbf{Z}$-graded ring. Show that there is an isomorphism of graded rings $\mathcal{A}_\bullet^{\mathbf{C}} \simeq \mathbf{Z}[\alpha]$ where α is of degree two.

(3) For $Y \subset X$ a closed subspace of a topological space X we have defined the groups $L_n(X, Y)$ in class. Show that the natural map $L_n(X, Y) \to L_{n+1}(X, Y)$ is an isomorphism.

Problem 2. *The second Steifel–Whitney class*

In this problem you will show that a manifold M admits a spin structure if and only if $w_2(M) = 0$.

(1) Fix a metric and an orientation on M. Pick a good cover $\mathcal{U} = \{U_\alpha\}$ of M which trivializes the frame bundle Fr_M^{SO} and suppose $\{g_{\alpha\beta} \colon U_{\alpha\beta} \to SO(n)\}$ are the transition functions. Argue that there exists lifts $\{\widetilde{g}_{\alpha\beta} \colon U_{\alpha\beta} \to Spin(n)\}$ such that $\widetilde{g}_{\alpha\alpha} = \mathbb{1}$ and $\widetilde{g}_{\alpha\beta} = g_{\beta\alpha}^{-1}$.

(2) Define $\epsilon_{\alpha\beta\gamma} = \widetilde{g}_{\alpha\gamma}\widetilde{g}_{\beta\gamma}\widetilde{g}_{\gamma\beta}$. Show that $\epsilon_{\alpha\beta\gamma}(p) \in \ker(Spin(n) \to SO(n)) = \mathbf{Z}/2$.

(3) Check that $\{\epsilon_{\alpha\beta\gamma}\}$ defines a Čech cocycle.

(4) Show that this cocycle is independent of the choice of a lift in (a). Thus, $\{\epsilon_{\alpha\beta\gamma}\}$ gives a well-defined element

$$w_2(M) \stackrel{\text{def}}{=} [\{\epsilon_{\alpha\beta\gamma}\}] \in \check{H}^2(M; \mathbf{Z}/2) \simeq H^2(M; \mathbf{Z}/2)$$

which vanishes if and only if M admits a spin structure.

Problem 3. *Spin structures on Kahler manifolds*

(1) Recall what an almost complex manifold, what a complex manifold is, and what the canonical line bundle is.

(2) Endowing $\mathbf{R}^{2n}$ with its standard almost complex structure defines a homomorphism $i\colon U(n) \to SO(2n)$. Show that the induced map

$$i^*\colon H^2(SO(2n);\mathbf{Z}/2) \to H^2(U(n);\mathbf{Z}/2) \tag{2}$$

is injective.

(3) Consider the determinant map $\det\colon U(n) \to U(1)$. Show that

$$\det^*\colon H^2(U(1);\mathbf{Z}/2) \to H^2(U(n);\mathbf{Z}/2) \tag{3}$$

is injective.

(4) Argue that spin structures on an almost complex manifold X are in bijective correspondence with double coverings of the $U(1)$ bundle $\det(Fr_X^{SO(2n)})$ which restrict to the squaring map $U(1) \to U(1), z \mapsto z^2$ on each fiber.

(5) Further, show that on a complex manifold X spin structures are in bijective correspondence with the set of isomorphism classes of holomorphic line bundles L on X which satisfy $L^{\otimes 2} \simeq K_X$. Such line bundles L are called "square-roots" of the canonical bundle.