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Linear field equations on self-dual spaces

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In a Riemannian context, a description is given of the Penrose correspondence between solutions of the anti-self-dual zero rest-mass field equations in a self-dual Yang–Mills background on a self-dual space X, and the sheaf cohomology groups $H^1(Z, \mathcal{O}F(n))$, for $n \leqslant -2$, of its twistor space Z. The case $n = -2$ is fundamental for the construction of instantons on Euclidean space. It is further shown how $H^1(Z, \mathcal{O}F(-1))$ corresponds to solutions of the self-dual Dirac equation, and an interpretation for $H^1(Z, \mathcal{O}F(n))$, for $n \geqslant 0$, is given in terms of the cohomology of an elliptic complex on X.

INTRODUCTION

In an earlier paper (Atiyah, Hitchin & Singer 1978) we gave an account of the ideas of R. Penrose connecting the Riemannian geometry of certain four-dimensional spaces with complex three-dimensional geometry. We were basically interested there in solutions to the self-dual Yang-Mills equations and we showed, among other things, how using the ideas of Penrose and Ward such solutions could be interpreted in terms of holomorphic vector bundles. Since then, the methods of modern algebraic geometry have provided a complete solution to the problem of finding all self-dual Yang–Mills fields on Euclidean 4-space (Atiyah, Hitchin, Drinfeld & Manin 1978). The crucial step in this work was the vanishing of a certain sheaf cohomology group: this was proved by interpreting it as the space of solutions of the conformally invariant Laplace equation on Euclidean space. This interpretation, without the Yang–Mills field, has been a fundamental part of the Penrose twistor programme for some time and the purpose of the present paper is to describe this transformation within the general context of self-duality.

There are several differential equations whose solutions admit a holomorphic interpretation: the conformally invariant Laplace equation, the Dirac equation and the anti-self-dual zero rest-mass field equations for higher spin, including the Maxwell equations. The unifying feature of all these is their conformal invariance. In § 1 it is shown how to describe this invariance for first order operators in an algebraic way. The essential point is that the first order differential equations we are interested in are defined by first derivatives of the conformal structure. Since it requires second derivatives and curvature to distinguish any conformal structure from the flat one, we obtain a large 'internal symmetry group' on the 4-dimensional manifold consisting of the conformal transformations of the 4-sphere S^4 that leave fixed a point, and it is the invariant subspaces of this group that characterize the equations. The conformal invariance of the zero-rest mass field equations was shown by Penrose (1965) but the point of view we adopt here is due to Fegan (1976). The Laplacian is of course a second order operator and cannot be characterized by conformal invariance in the same way but we do show that it annihilates the inner product of solutions to two first order equations – the Dirac equation and the twistor equation – and this it turns out is enough to identify it.

In § 2 we recall the construction of the three-dimensional complex twistor space Z of a four-manifold X with a self-dual conformal structure (i.e. such that the anti-self-dual part W_- of the Weyl tensor vanishes), and we show how solutions of a certain first-order overdetermined system of equations on X (the conformal Killing spinors) correspond to the sheaf cohomology groups $H^0(Z, \mathcal{O}(m))$.

In the third section we prove the main theorem: that the cohomology groups $H^1(Z, \mathcal{O}F(-m-2))$ for $m \geqslant 0$ correspond precisely to solutions of the anti-self-dual zero rest-mass field equations of spin $\frac{1}{2}m$ in a self-dual Yang–Mills background corresponding to the holomorphic vector bundle F on Z. In the case of $X = S^4$, proofs of the theorem for $m = 0$ for Dolbeault cohomology have been given by Drinfeld & Manin (1978) and Rawnsley (1979). Douady (1978) has also given a spectral sequence argument. Although the Dolbeault approach is used for part of the present proof, the essential point, which is all that was necessary for the case of the four-sphere, is provided by an algebraic argument. We show, by using the technique of formal neighbourhoods, that the cohomology group corresponds to solutions of some differential equation, and then identify it by conformal invariance. The use of formal neighbourhoods has much to recommend it in this context since it avoids local computations which, by the very global nature of the Penrose transform, are awkward to perform.

It should be noted that the original idea behind the description of a Yang–Mills field in terms of a holomorphic vector bundle given by Ward (1977) was to represent in complex analytic terms the interaction of such zero rest-mass fields in a Yang–Mills background. Useful references for matters related to those described here are Lerner & Sommers (1979) and Wells (1979).

Finally in § 4 it is shown how the cohomology groups $H^1(Z, \mathcal{O}F(-1))$ correspond to solutions of the self-dual Dirac equation in a self-dual Yang–Mills background and that in general the groups $H^1(Z, \mathcal{O}F(m))$ $m \geqslant 0$ are interpreted in terms of first cohomology groups of certain conformally invariant elliptic complexes on X.

The present paper may be regarded as a sequel to (Atiyah, Hitchin & Singer 1978) and much of the notation and terminology is taken directly from this work. In the text references to it are abbreviated to (A.H.S). The author wishes to thank M. F. Atiyah for many conversations, and this paper consists in large part of joint work.

1. Conformal invariance

The two basic equations with which we shall be concerned are:

(i) Laplace's equation $\sum_i \frac{\partial^2 f}{\partial x_i^2} = 0$ and

(ii) the Dirac equation $\sum_i e_i \cdot \frac{\partial \phi}{\partial x_i} = 0$,

where f is a scalar function and ϕ a spinor field. These can be generalized in two ways: first we can allow f and ϕ to take their values in vector bundles and secondly we can define the equations on a curved background. To achieve the former, we just replace ordinary derivatives by covariant derivatives and on a *Riemannian* manifold one could do the same for the latter using the Riemannian connection, but in what follows we shall need to exploit the conformal invariance of (i) and (ii), so we shall first consider how to define a differential equation in general terms without using a connection.

To do this we use the language of jets. If E is a vector bundle over a manifold X, then the space of k-jets at $x \in X$ of sections of E is the space of Taylor expansions up to the kth order at x of sections of E. As x varies over X we get a vector bundle $J_k(E)$, the k-jet bundle of E, and to each section $s \in \Gamma(E)$ we associate its k-jet $j_k(s) \in \Gamma(J_k(E))$. A linear differential operator $\mathrm{D}: \Gamma(E) \to \Gamma(E)$ of order k is now just a vector bundle homomorphism from $J_k(E)$ to F, since $\mathrm{D}s$ is linear in the first $(k+1)$ terms of the Taylor series of s at each point. A kth order differential equation $\mathrm{D}s = 0$ is defined by the family of subspaces $R \subset J_k(E)$ which is the kernel of this homomorphism.

The k-jet determines the $(k-1)$-jet, and we get an exact sequence of vector bundles:

$$0 \to S^kT^* \otimes E \to J_k(E) \to J_{k-1}(E) \to 0, \tag{1.1}$$

where S^kT^* is the kth symmetric power of the cotangent bundle. This says that if the first k terms of the Taylor series vanish, the next term is invariantly defined as a homogeneous polynomial of degree k. The composite homomorphism

$$S^kT^* \otimes E \to J_k(E) \to F,$$

defined by D is the *symbol*, or highest order part, of the operator D. Clearly from (1.1) two operators with the same symbol differ by a lower order operator.

As an example, suppose the bundle E has a connection. This defines a first order operator $\nabla: \Gamma(E) \to \Gamma(T^* \otimes E)$, the covariant derivative. The corresponding homomorphism from $J_1(E)$ to $T^* \otimes E$ is just a splitting of the exact sequence (1.1) for $k = 1$:

$$0 \to T^* \otimes E \leftrightarrows J_1(E) \to E \to 0.$$

Suppose now that X is an oriented n-manifold with a conformal structure. We can define this as an equivalence class of Riemannian metrics where $g_1 \sim g_2$ if $g_1 = e^f g_2$, as a positive definite inner product on the bundle $T \otimes \Omega^{1/n}$ where Ω is the bundle of densities, or in the language of G-structures as a reduction of the structure group of the cotangent bundle to $CO(n) = \{aA : a \in \mathbb{R}^+, A \in SO(n)\}$. We may think of $CO(n)$ as the group of automorphisms of the 0-jet of the conformal structure at each point x. The 'internal symmetry group' $CO(n)$ allows us to define associated vector bundles corresponding to any representation ρ of $CO(n)$. Note that an irreducible representation of $CO(n)$ is given by an irreducible representation of $SO(n)$ together with a conformal weight $w \in \mathbb{R}$ so that $a \in CO(n)$ acts as a^w.

Instead of considering just $CO(n)$, we can look at the group of automorphisms of the 1-jet of the conformal structure at X. Since it takes curvature and second derivatives to distinguish any two conformal structures, this group is the same as for the flat case: it is just the group of conformal transformations of the n-sphere S^n which leave fixed a point. Taking this point as infinity, it is easy to see that this is the group $CE(n)$ generated by the Euclidean transformations of $\mathbb{R}^n$ and dilations: a semi-direct product of $CO(n)$ with the translation group $\mathbb{R}^n$.

Now suppose E is a bundle over X associated to a representation space $\boldsymbol{E}$ of $CO(n)$, then $J_1(E)_x$, the space of 1-jets of sections of E at x, is defined by the 1-jet of the conformal structure and thus is acted upon by $CE(n)$. A subspace $R_x \subset J_1(E)_x$ which is invariant by $CE(n)$ will then define a conformally invariant differential equation as x varies. We shall use this algebraic approach to characterize the Dirac equation and the corresponding zero rest-mass field equations for higher spin. This is the point of view adopted by Fegan (1976) and details for what follows may be found in his paper.

The first thing we need to know is the action of $CE(n)$ on $J_1(E)_x$. If we choose a decomposition of $CE(n)$ into a semi-direct product $\mathbb{R}^n . CO(n)$, then $J_1(E)_x$ splits as a $CO(n)$ module (essentially the splitting given by the Riemannian connection of a metric within the conformal class) into a direct sum $\boldsymbol{E} \oplus (\boldsymbol{E} \otimes \mathbb{R}^n)$ and the translation part maps $\boldsymbol{E}$ to $\boldsymbol{E} \otimes \mathbb{R}^n$ as follows:

$$v \mapsto \sum_i \rho(x \otimes e_i^* - e_i \otimes x^*)\, v \otimes e_i + wv \otimes x$$

where $v \in \boldsymbol{E}$ the representation space of $CO(n)$, $x \in \mathbb{R}^n$, $\{e_i\}$ is an orthonormal basis $\mathbb{R}^n$, ρ: $so(n) \to \operatorname{End} \boldsymbol{E}$ is the representation on the Lie algebra and w the conformal weight. If $\boldsymbol{E}$ is an irreducible representation of $SO(n)$, it follows that any $CE(n)$ submodule of $J_1(E)_x$ which projects onto E_x must be of the form

$$\boldsymbol{R} = \boldsymbol{E} \oplus \operatorname{Im}(B + wI)$$

where B: $\boldsymbol{E} \otimes \mathbb{R}^n \to \boldsymbol{E} \otimes \mathbb{R}^n$ is given by

$$B(v \otimes x) = \sum_i \rho(x \otimes e_i^* - e_i \otimes x^*)\, v \otimes e_i.$$

Thus we get a proper invariant subspace only if the conformal weight w is the negative of an eigenvalue of B and then the corresponding differential operator maps sections of E to sections of the bundle corresponding to the null-space of $B + wI$. B is clearly an $SO(n)$-invariant map and can be written in terms of Casimir operators:

$$\tfrac{1}{2}B = C(\boldsymbol{E}) \otimes 1 + 1 \otimes C(\mathbb{R}^n) - C(\boldsymbol{E} \otimes \mathbb{R}^n), \tag{1.2}$$

where $C(\boldsymbol{E}) = \Sigma\rho(X_i)^2$, $\{X_i\}$ being an orthonormal basis of $so(n)$ relative to the Killing form. Since the value of the Casimir on an irreducible representation is well known, one can immediately inspect the possibilities.

Let us specialize (1.2) to the case $n = 4$ which concerns us here. We have an isomorphism $\mathrm{Spin}\,(4) \cong SU(2) \times SU(2)$ and the basic spin representations $\boldsymbol{V}_+$, $\boldsymbol{V}_-$ of the two factors. It is well known that every complex irreducible representation of Spin (4) is of the form $S^m\boldsymbol{V}_+ \otimes S^n\boldsymbol{V}_-$ where $S^m\boldsymbol{V}$ denotes the mth symmetric power of $\boldsymbol{V}$. This is an $(m+1)$-dimensional representation with Casimir operator $-m(m+2)/8$. We shall abbreviate $S^m_-\boldsymbol{V}_+$, $S^m\boldsymbol{V}_-$ to $\boldsymbol{S}^m_+$, $\boldsymbol{S}^m_-$ in what follows. Note that $\mathbb{R}^4 = \boldsymbol{S}_+ \otimes \boldsymbol{S}_-$, so $C(\mathbb{R}^4) = -2\,.\,\frac{3}{8} = -\frac{3}{4}$.

Examples

1. Let $\boldsymbol{E}$ be the trivial 1-dimensional representation of $SO(n)$, then from (1.2) $B = 0$, hence $w = 0$ and the only invariant first order differential operator is the usual differential $\mathrm{d}f$ of a function.

2. Let $\boldsymbol{E} = \mathbb{R}^4 = \boldsymbol{S}_+ \otimes \boldsymbol{S}_-$. Then $\mathbb{R}^4 \otimes \mathbb{R}^4$ splits into four irreducible components $(\boldsymbol{S}^2_+ \otimes \boldsymbol{S}^2_-, \boldsymbol{S}^2_+, \boldsymbol{S}^2_-, \mathbf{1})$ with Casimir operators $(-2, -1, -1, 0)$ and we find four conformally invariant operators of weight $(-1, 1, 1, 3)$ respectively. The first is the operator whose null-space consists of conformal vector fields, and the others are all part of the de Rham complex: the operators $\mathrm{d} + *\mathrm{d}$, $\mathrm{d} - *\mathrm{d}$ on 1-forms, and d on 3-forms.

3. Take $\boldsymbol{E} = \boldsymbol{S}_-$, then $\boldsymbol{S}_- \otimes \mathbb{R}^4 = \boldsymbol{S}_- \otimes \boldsymbol{S}_- \otimes \boldsymbol{S}_+$ decomposes into only two irreducible components: $\boldsymbol{S}^2_- \otimes \boldsymbol{S}_+$ with Casimir $-\frac{7}{8}$ and $\boldsymbol{S}_+$ with value $-\frac{3}{8}$, so we get two operators:

(i) $\mathrm{D}\colon \Gamma(S_-) \to \Gamma(S_+)$, the Dirac operator with conformal weight $\frac{3}{2}$ and
(ii) $\bar{\mathrm{D}}\colon \Gamma(S_-) \to \Gamma(S^2_- \otimes S_+)$, the twistor operator with conformal weight $-\frac{1}{2}$.

4. If $\boldsymbol{E} = \boldsymbol{S}^m$ we get two analogous operators:

(i) $\mathrm{D}_m\colon \Gamma(S^m_-) \to \Gamma(S^{m-1}_- \otimes S_+)$ with weight $\frac{1}{2}(m+2)$: this is the operator whose null-space consists of anti-self-dual zero rest-mass fields of spin $\frac{1}{2}m$, and is really part of the extended Dirac operator $\mathrm{D}\colon \Gamma(S_- \otimes S^{m-1}_-) \to \Gamma(S_+ \otimes S^{m-1}_-)$ if we fix a metric. In particular for $m = 2$, $\mathrm{D}_2\colon \Gamma(S^2_-) \to \Gamma(S_- \otimes S_+)$ has weight 2 and is just the exterior derivative d restricted to anti-self-dual 2-forms. Its null-space consists therefore of anti-self-dual Maxwell fields.

(ii) $\bar{\mathrm{D}}_m\colon \Gamma(S^m_-) \to \Gamma(S^{m+1}_- \otimes S_+)$ of weight $-\frac{1}{2}m$.
The null space of this operator consists of anti-self-dual conformal Killing spinors.

There are of course equivalent operators on the bundles S^m_+.

We have now characterized the Dirac equation $\mathrm{D}\phi = 0$ as the unique $CE(4)$-invariant subspace of $J_1(S_-)$ at each point which projects onto S_-, where S_- has conformal weight $\frac{3}{2}$. We can now extend this characterization to a Yang–Mills background, that is, we are given a principal G-bundle with connection, a representation $\boldsymbol{E}$ of G and an associated bundle E. Then $J_1(S_- \otimes E)_x$ is a $CE(4) \times G$-module and if E is an irreducible representation, there is a unique invariant subspace which defines the extended Dirac operator $\mathrm{D}: \Gamma(S_- \otimes E) \to \Gamma(S_+ \otimes E)$.

We can immediately deduce from the conformal invariance some remarkable properties of solutions to the equations $\mathrm{D}_m \phi = 0$ and $\bar{\mathrm{D}}_m \psi = 0$. First suppose $\bar{\mathrm{D}}_1 \phi = 0$ and $\bar{\mathrm{D}}_m \psi = 0$. There is a natural map (symmetrization) from $S_- \otimes S_-^m$ to S_-^{m+1} and since ϕ has weight $-\frac{1}{2}$ and ψ weight $-\frac{1}{2}m$ their symmetrized product has weight $-\frac{1}{2}(m+1)$. Now under the induced map from $J_1(S_-) \otimes J_1(S_-^m)$ to $J_1(S_-^{m+1})$ the invariant subspaces $\bar{R}_1$, $\bar{R}_m$ generate an invariant subspace of $J_1(S_-^{m+1})$ and this, by what we have seen, must be $\bar{R}_{m+1}$ or the whole space. It is in fact a proper subspace, as can be seen by considering all spaces as Spin (4)-modules and so must be $\bar{R}_{m+1}$: in other words the symmetrized product of ϕ and ψ satisfies the equation $\bar{\mathrm{D}}_{m+1} \phi = 0$. We can do the same for $\bar{R}_m$ and $\bar{R}_n$ and then in algebraic terms, we see that this product gives $\bigoplus_m \ker \bar{\mathrm{D}}_m$ the structure of a commutative ring. This of course may be trivial: there are integrability conditions to be satisfied in order for solutions to $\bar{\mathrm{D}}_m \phi = 0$ to exist at all.

Now take a solution to the twistor equation $\bar{\mathrm{D}}_1 \phi = 0$ and one of the zero rest-mass field equation $\bar{\mathrm{D}}_m \psi = 0$. There is a natural map (contraction with respect to a symplectic form) from $S_- \otimes S_-^m$ to S_-^{m-1}. Since ϕ has weight $-\frac{1}{2}$, and ψ weight $\frac{1}{2}(m+2)$, their contraction has weight $\frac{1}{2}(m+1)$ and a similar argument to the above one shows that the contraction θ of ϕ and ψ satisfies the field equation $\mathrm{D}_{m-1} \theta = 0$ (Isham *et al.* 1979 p. 348).

All that we have said so far is valid on an arbitrary conformal 4-manifold. Our main concern, however, is with a self-dual manifold (A.H.S., § 1) i.e. one for which $W_- = 0$ and a self-dual Yang–Mills field thereon, i.e. a connection whose curvature $F \in \Gamma(\mathfrak{g} \otimes \Lambda_+^2)$. In this situation, if we fix a Riemannian metric within the conformal structure, the second order operator $\mathrm{D}_m^* \mathrm{D}_m$(where D_m^* is the formal adjoint of D_m) has a Weitzenböck decomposition (A.H.S., § 6) involving only the scalar curvature R: in fact we may write,

$$\mathrm{D}_m^* \mathrm{D}_m = \nabla^* \nabla + \tfrac{1}{12}(m+2)\, R. \tag{1.3}$$

(The case $m = 1$ is the formula of Lichnerowicz (1963); the case $m = 2$ was used in (A.H.S., § 6) in computing the moduli of instantons.) We should note that these second order operators are not conformally invariant. However, the formula (1.3) and the properties of zero rest-mass fields under contraction with a solution of $\bar{D}_1 \phi = 0$ do lead together to a conformally invariant second order operator which in some sense completes the set of operators D_m, $m \geqslant 1$.

Suppose $\bar{\mathrm{D}}_1 \phi = 0$ and $\mathrm{D}_1 \psi = 0$, then we can contract ϕ and ψ using the symplectic form on S_- and obtain a scalar $f = \langle \phi, \psi \rangle$ of conformal weight 1, i.e. a $\frac{1}{4}$-density. From Example 1, this satisfies no conformally invariant first order equation: it does however satisfy a second order one. If we fix a metric, then

$$\nabla^2 f = \langle \nabla^2 \phi, \psi \rangle + \langle \phi, \nabla^2 \psi \rangle \in \Gamma(S^2 T^*)$$

since $\nabla \phi$ and $\nabla \psi$ are in orthogonal subbundles of $S_- \otimes T^*$, and so

$$\nabla^* \nabla f = \langle \nabla^* \nabla \phi, \psi \rangle + \langle \phi, \nabla^* \nabla \psi \rangle.$$

But from (1.3) $\nabla^*\nabla\psi = -\frac{1}{4}R\psi$ and a similar formula for the twistor equation gives $\nabla^*\nabla\phi = \frac{1}{12}R\phi$, hence

$$(\nabla^*\nabla + \tfrac{1}{6}R)f = 0. \tag{1.4}$$

We denote this operator (the *Laplacian* acting on scalars of weight 1) by D_0 (cf. (1.3)). Its conformal invariance may be seen in the following geometrical way. Given a conformal structure, all one needs to define a metric is a scale, or in other words a non-vanishing scalar density. Thus if f is a scalar of weight 1, and non-zero in some neighbourhood, we define $\mathrm{D}_0 f$ as $\frac{1}{6}Rf^3$, where R is the scalar curvature of the corresponding metric. The formula for the change in scalar curvature under a conformal change of metric (Eisenhart 1925)

$$\nabla^*\nabla u + \frac{(n-2)}{4(n-1)}Ru = \frac{(n-2)}{4(n-1)}\tilde{R}u^{(n+2)/(n-2)}$$

$$u^{4/(n-2)}f = \tilde{g}$$

shows that for $n = 4$ this really is the linear differential operator (1.4). Alternatively, one can prove the conformal invariance directly and, by replacing ordinary derivatives by covariant derivatives, extend it to an operator in a Yang–Mills background:

$$\mathrm{D}_0 \colon \Gamma(E) \rightarrow \Gamma(E)$$

from sections of E of weight 1 to sections of weight 3. Note that D_0 is a symmetric operator, i.e.

$$\int (\mathrm{D}_0\phi, \psi) = \int (\phi, \mathrm{D}_0\psi)$$

(the inner product is of weight 4 – a density – and can therefore be integrated).

We might note in passing that by taking the tracefree Ricci tensor of the metric determined by a scalar density we obtain a conformally invariant overdetermined 2nd-order operator

$$\bar{\Delta} \colon \Gamma(1) \rightarrow \Gamma(S^2_+ \otimes S^2_-)$$

acting on scalars of weight -1. Thus whereas a non-vanishing solution of $D_0 f = 0$ defines a metric with zero scalar curvature, a solution of $\bar{\Delta}f = 0$ defines an Einstein metric.

The field equations we are concerned with here are, then, the anti-self-dual zero rest-mass field equations in a self-dual Yang–Mills background:

$$\mathrm{D}_m\phi = 0,\ \phi \in \Gamma(S^m_- \otimes E) \quad \text{of weight} \quad \tfrac{1}{2}(m+2) \quad (0 \leqslant m < \infty).$$

By using their conformal invariance properties and the relation with solutions of the twistor equation we shall give them a holomorphic interpretation under the Penrose transform.

2. Twistors

Before considering the field equations in a holomorphic context, let us recall the definition of the complex structure of the *twistor space* of a self-dual manifold, as given in (A.H.S., § 4). This is based on the problem of finding solutions to the twistor equation $\overline{\mathrm{D}}_1\phi = 0$: one takes the bundle $S_-(=V_-)$ of weight $-\frac{1}{2}$ and considers the total space $S_-^*\backslash 0$ of the dual bundle minus its zero section. A section $\phi \in \Gamma(S_-)$ then defines a complex-valued function $\phi^\vee$ on $S_-^*\backslash 0$ and ϕ satisfies the twistor equation $\overline{\mathrm{D}}_1\phi = 0$ iff $\mathrm{d}\phi^\vee$ lies in a certain subbundle $V(\overline{\mathrm{D}})$ of $T_c^*(S_-^*\backslash 0)$. This bundle, if $W_- = 0$, is involutive and defines a complex structure on $S_-^*\backslash 0$, that is consists of the subbundle of $(1,0)$ forms. In other words, $\overline{\mathrm{D}}\phi = 0$ if and only if $\bar\partial\phi^\vee = 0$ and so $\phi^\vee$ is a holomorphic function. Since $\phi^\vee$ is linear in the fibre variables of $S_-^*\backslash 0$, it defines a holomorphic section of the bundle H over the projective twistor space $P(S_-^*) = Z$, where $S_-^*\backslash 0$ is the principal bundle of H.

We thus have immediately a holomorphic interpretation of solutions to the twistor equation: these are precisely the holomorphic sections of H over Z, which we denote by $H^0(Z, \mathcal{O}(1))$, the 0-th cohomology group of the sheaf of germs of holomorphic sections of H. With the same approach, it is easy to see that a section $\phi \in \Gamma(S_-^m)$ of weight $-\frac{1}{2}m$ defines a function $\phi^\vee$ on $S_-^*\backslash 0$, a homogeneous polynomial of degree m in the fibres, and that $\bar\partial\phi^\vee = 0$ if and only if $\overline{\mathrm{D}}_m\phi = 0$. Hence the space of solutions of $\overline{\mathrm{D}}_m\phi = 0$ corresponds to the space $H^0(Z, \mathcal{O}(m))$ of holomorphic sections of H^m $(= H \otimes H \ldots \otimes H)$ over Z. Thus in the self-dual case, the algebraic structure on $\oplus \ker \overline{\mathrm{D}}_m$ referred to in § 1 is just the ring structure on $\oplus\, H^0(Z, \mathcal{O}(m))$ obtained by taking products of holomorphic sections of line bundles.

We obtain a slightly different viewpoint if we adopt a holomorphic approach and see Z as a complex 3-manifold with a 4-parameter family X^c of projective lines (the real lines of this family, as defined in (A.H.S., § 4), are the fibres of $Z = P(S_-^*) \to X$). If we now start with an element $s \in H^0(Z, \mathcal{O}(m))$ we may restrict it to each line, and since the bundle H restricts to the standard positive bundle on each P_1 we have for each $x \in X$,

$$Ts(x) \in H^0(P(S_-^*)_x, \mathcal{O}(1)) \cong (S_-)_x$$

which gives a section $Ts \in \Gamma(S_-)$ of weight $-\frac{1}{2}$ satisfying the twistor equation.

This transform, or restriction map, may also be applied to higher cohomology groups. Since P_1 is 1-dimensional and $H^1(P_1, \mathcal{O}(m)) = 0$ if $m \geqslant -1$, the only non-zero possibilities are $H^1(Z, \mathcal{O}(-m-2))$, where $m \geqslant 0$. We shall show next that under the restriction map these define precisely the solutions of $\mathrm{D}_m\phi = 0$.

3. Field equations and cohomology

Let us consider S_- as a $CO(4)$ module of weight $-\frac{1}{2}$. then $H^1(P(S_-^*), \mathcal{O}(-m-2))$ is of weight $(-m-2).(-\frac{1}{2}) = \frac{1}{2}(m+2)$ and by Serre duality is isomorphic to S_-^m. We thus have by restriction a transform

$$T\colon H^1(Z, \mathcal{O}(-m-2)) \to \Gamma(S_-^m)$$

to sections of weight $\frac{1}{2}(m+2)$.

Suppose now E is a bundle on X with a self-dual G-connexion, then from (A.H.S., § 5) we know that its pull-back to Z, F, has a holomorphic structure and is holomorphically trivial on the real lines. We thus obtain a corresponding transform

$$T\colon H^1(Z, \mathcal{O}(F(-m-2))) \to \Gamma(S_-^m \otimes E).$$

With this notation we shall prove the following theorem:

Theorem (3.1). *Let X be a self-dual space and E a bundle over X with self-dual connection, then*

$$T\colon H^1(Z, \mathcal{O}F(-m-2)) \to \Gamma(S_-^m \otimes E)$$

defines an isomorphism onto the space of solutions to the field equation $\mathrm{D}_m\phi = 0$, $m \geqslant 0$.

Proof. There are three things to prove: first that T is injective, secondly that $T\alpha$ satisfies the field equation, and finally that every solution of $\mathrm{D}_m\phi = 0$ is obtained in this way. For the main application of this (Atiyah *et al.* 1978) the first part was true from general algebraic considerations, but in the present context where Z is neither compact nor algebraic we shall use a different approach, based on that of Rawnsley (1979).

We use the Dolbeault representative of a class $\alpha \in H^1(F(-m-2))$, that is we represent α by a $\bar{\partial}$-closed $(0,1)$ form $\omega \in A^{0,1}(F \otimes H^{-m-2})$. If $T\alpha = 0$ then on each fibre Z_x, $\omega = \bar{\partial}s_x$ for some $s_x \in \Gamma(Z_x, F \otimes H^{-m-2})$. However, since F is holomorphically trivial on Z_x and $H^0(P_1, \mathcal{O}(-m-2)) = 0$ for $m \geqslant 0$, s_x is unique and by elliptic regularity we have a smooth section s of $F \otimes H^{-m-2}$ such that $\omega - \bar{\partial}s$ restricts to zero on each fibre, as a 1-form.

We now pull back this form to $\tilde{\omega}$ on $S_-^*\backslash 0$ and express it locally in terms of a basis of $(0,1)$ forms. From the construction of the complex structure on $S_-^*\backslash 0$, the space of $(1,0)$ forms is locally spanned by θ_1, θ_2, $\sigma_1^\vee$, $\sigma_2^\vee$ (A.H.S., § 4) where

$$\theta_\alpha = \mathrm{d}\lambda_\alpha - \Sigma\omega_{\alpha\beta}\lambda_\beta, \quad \sigma_\alpha^\vee = \Sigma\lambda_\beta \langle e_i . \psi_\alpha, \phi_\beta\rangle e_i$$

and $\{\psi_\alpha\}$, $\{\phi_\alpha\}$, $\{e_i\}$ are local bases of S_+, S_- and T^* respectively. Since $\tilde{\omega}$ restricts to zero on each fibre,

$$\tilde{\omega} = \sum_\alpha s_\alpha \otimes \bar{\sigma}_\alpha^\vee, \quad s_\alpha \in \Gamma(F \otimes H^{-m-2}). \tag{3.2}$$

On the other hand $\bar{\partial}\tilde{\omega} = 0$, so

$$0 = \sum_\alpha \bar{\partial}s_\alpha \wedge \bar{\sigma}_\alpha^\vee + \sum_\alpha s_\alpha \otimes \bar{\partial}\bar{\sigma}_\alpha^\vee. \tag{3.3}$$

But now

$$d\bar{\sigma}^\vee_\alpha = \sum_{i,\beta} \langle e_i \cdot \psi_\alpha, \phi_\beta \rangle^- \bar{\theta}_\beta \wedge e_i + \sum_{i,j,\beta} \langle e_i \cdot \nabla_j \psi_\alpha, \phi_\beta \rangle^- \bar{\lambda}_\beta e_j \wedge e_i$$

We may choose $\{\psi_\alpha\}$ such that, at a fixed point $x \in X$, $\nabla \psi_\alpha = 0$ and then

$$\begin{aligned}\bar{\partial}\bar{\sigma}^\vee_\alpha &= \Sigma \langle e_i \cdot \psi_\alpha, \phi_\beta \rangle^- \bar{\theta}_\beta \wedge (e_i)^{0,1} \quad \text{on} \quad (S^*_-)_x \\ &= \Sigma \frac{\langle e_i \cdot \psi_\alpha, \psi_\beta \rangle^- \bar{\theta}_\beta \wedge \langle e_i \cdot \psi_\gamma, \phi \rangle \bar{\sigma}^\vee_\gamma}{\langle e_j \cdot \psi_\gamma, \phi \rangle^- \langle e_j \cdot \psi_\gamma, \phi \rangle} \\ &= \Sigma \frac{\lambda_\beta}{|\lambda|^2} \bar{\theta}_\beta \wedge \bar{\sigma}^\vee_\alpha,\end{aligned}$$

Hence from (3.3)

$$0 = \Sigma\, \bar{\partial} s_\alpha \wedge \bar{\sigma}^\vee_\alpha + \Sigma s_\alpha \otimes \frac{\lambda_\beta}{|\lambda|^2} \bar{\theta}_\beta \wedge \bar{\sigma}^\vee_\alpha$$

and $\bar{\partial} s_\alpha + \Sigma s_\alpha \otimes \lambda_\beta \, d\bar{\lambda}_\beta / |\lambda|^2 = 0$ on each fibre, so $\bar{\partial}(|\lambda|^2 s_\alpha) = 0$. But in (3.2) since $\tilde{\omega}$ is pulled back from $P(S^*_-)$ and $\sigma^\vee_\alpha$ is linear in λ and furthermore F is holomorphically trivial on each fibre, each s_α may be regarded as a vector valued function of $\phi \in (S^*_- \backslash 0)_x$ satisfying the homogeneity equation

$$s_\alpha(\lambda\phi) = \lambda^{-m-2} \bar{\lambda}^{-1} s_\alpha(\phi)$$

thus $|\lambda|^2 s_\alpha$ is homogeneous of degree $-(m+1)$ and holomorphic, so must vanish for $m \geqslant 0$.

Hence $\tilde{\omega} = 0$, $\omega = \bar{\partial} s$ and the cohomology class α is trivial i.e. $T\alpha = 0$ implies $\alpha = 0$.

We must now show that $T\alpha$ satisfies the differential equation $\mathrm{D}_m T\alpha = 0$. To do this, we revert to the terminology of jets, but in a more algebraic geometrical language.

If U is a complex manifold, then at a point $u \in U$, we may consider the ideal sheaf $\mathscr{I}_u$ of germs of local holomorphic functions that vanish at u, contained in the sheaf $\mathcal{O}$ of all local holomorphic functions. The quotient sheaf $\mathcal{O}/\mathscr{I}_u^{k+1}$ is then supported on u and its stalk consists there of all Taylor series up to order k of functions at u. Similarly, if E is a holomorphic vector bundle on U, then the stalk of the sheaf $\mathcal{O}(E)/\mathscr{I}_u^{k+1} \otimes \mathcal{O}(E)$ is the space of k-jets of holomorphic sections of E at u.

Instead of considering a *point* $u \in U$, we can consider a *subvariety* $V \subset U$ and its ideal sheaf $\mathscr{I}_V$. The sheaf $\mathcal{O}/\mathscr{I}_V^{k+1}$ is then supported on V and we can think of V as a space with an enlarged ring of holomorphic functions that captures the information contained in the first k normal derivatives of any object defined in a neighbourhood of V. The sheaf is called the *kth order formal neighbourhood* of V in U (Griffiths 1966).

This point of view is appropriate to the present situation because the *points* of X^c parametrize *lines* in Z and a variation of $x \in X^c$ in its kth order neighbourhood defines a variation of the corresponding line L_x in its neighbourhood. We make this more precise as follows.

The complex 4-manifold X^c of lines on the 3-manifold Z has a natural complex conformal structure (A.H.S., §4). Associated to the conformal structure we have a 5-dimensional complex manifold $Y = P(S_-^*)^c$, the projective spin bundle over X^c. The manifold Y plays a fundamental role in the Penrose transform from the complex point of view. It has two projections:

$$\begin{tikzcd} & Z \\ Y \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\ & X^c \end{tikzcd}$$

p_2 is the projection when Y is considered as the projective spin bundle of X^c, it is complex analytically locally trivial; the other projection p_1 occurs because the fibre over $x \in X^c$ is naturally identified with the corresponding line L_x in Z. Restricting p_1 to the part of Y over the *real* subvariety $X \subset X^c$ gives the identification $P(S_-^*) \cong Z$.

Let us consider the transform T in these terms. We start with the holomorphic bundle $W = F \otimes H^{-m-2}$ on Z where F is trivial on all real fibres Z_x. For each $x \in X$, $H^1(Z_x, \mathcal{O}(W))$ is then of constant rank and defines a bundle on X which we denote $\mathscr{H}^1(W)$ ($\cong S_-^m \otimes E$) and T associates to $\alpha \in H^1(Z, \mathcal{O}(W))$ a section of this bundle by restriction. We may, however, restrict to *any* line of the family X^c. So long as F is trivial on these lines (and this will be true by semi-continuity for all x in a neighbourhood of $X \subset X^c$ so henceforth we take X^c to be such a neighbourhood) we obtain a holomorphic vector bundle $\mathscr{H}^1(W)$ on X^c and a corresponding section whose restriction to the real points $X \subset X^c$ is $T\alpha$. This extension to complex lines is obtained by pulling back the bundle W on Z to $p_1^* W$ on Y and restricting to the fibres of p_2. The transform T now factors through the map p_1^*:

$$T: H^1(Z, \mathcal{O}(W)) \xrightarrow{p^*_1} H^1(Y, \mathcal{O}(\mathrm{p}_1^* W)) \to H^0(X^c, \mathcal{O}(\mathscr{H}^1(W))) \to \Gamma(X, S_-^m \otimes E).$$

Now the fibration p_2: $Y \to X^c$ is holomorphically locally trivial with compact fibre, so if $B \subset X^c$ is a small open ball (and thus holomorphically convex), it follows from the Leray spectral sequence that the map

$$H^1(p_2^{-1}(B), \mathcal{O}_Y(p_1^* W)) \to H^0(B, \mathcal{O}_X(\mathscr{H}^1(W))) \tag{3.4}$$

to holomorphic sections is an *isomorphism*. If instead of considering a finite neighbourhood B of $x \in X^c$ and $p_2^{-1}(B)$ of the line $L_x (= Y_x)$ in Y, we restrict to formal neighbourhoods, we obtain an isomorphism

$$H^1(L_x, \mathcal{O}_Y^k(p_1^* W)) \cong H^0(x, \mathcal{O}_X^k(\mathscr{H}^1(W))) \cong J_k(S_-^m \otimes E)_x, \tag{3.5}$$

where $\mathcal{O}_Y^k$ denotes the kth order neighbourhood sheaf of $L_x \subset Y$.

The k-jet of $T\alpha$ then lies in the image of

$$p_1^*: H^1(L_x, \mathcal{O}_Z^k(W)) \to H^1(L_x, \mathcal{O}_Y^k(p_1^* W)) = J_k(S_-^m \otimes E)_x \tag{3.6}$$

and it is by examining this map that we shall find the relation on the k-jets (i.e. the differential equation) satisfied by $T\alpha$. The main tool is the exact sequence of sheaves

$$0 \to \mathcal{O}_V(S^k N^* \otimes E) \to \mathcal{O}_U^k(E) \to \mathcal{O}_U^{k-1}(E) \to 0 \tag{3.7}$$

where N^* is the conormal bundle of $V \subset U$ and E is a holomorphic bundle over U. If we apply this sequence to $L_x \subset Y$ and the vector bundle $p_1^* W$ and take the exact cohomology sequence, we just obtain the exact sequence of jet bundles (1.1) for the bundle $S_-^m \otimes E$. This is essentially because the normal bundle is trivial. The normal bundle of $L_x \subset Z$ is however non-trivial: it is isomorphic to $(S_+^*)_x \otimes H$ (A.H.S., §4). Taking the cohomology sequences for the two formal neighbourhoods of L_x and comparing them, we get the following diagram:

$$\begin{array}{ccccccccc}
(S_+^k \otimes S_-^{m+k} \otimes E)_x & & & & & & & & \\
\wr\Vert & & & & & & & & \\
0 \to H^1(S^k N^* \otimes F(-m-2)) & \to & H^1(\mathcal{O}_Z^k F(-m-2)) & \to & H^1(\mathcal{O}_Z^{k-1} F(-m-2)) & \to & 0 \\
\downarrow{\scriptstyle \sigma^*} & & \downarrow{\scriptstyle p_1^*} & & \downarrow{\scriptstyle p_1^*} & & \\
0 \to (S^k T^* \otimes S_-^m \otimes E)_x & \to & J_k(S_-^m \otimes E)_x & \to & J_{k-1}(S_-^m \otimes E)_x & \to & 0
\end{array}$$

where the H^0 terms disappear because N^* and $F \otimes H^{-m-2}$ are both negative.

If $k = 0$, p_1^* is an isomorphism by definition. For all k, the map σ^* is injective, as can be seen by evaluating on tangent vectors and using the isomorphism $T_x \cong H^0(L_x, \mathcal{O}(N))$. Hence by induction p_1^* is injective for all k and maps $H^1(\mathcal{O}_Z^k F(-m-2))$ into some subspace of $J_k(S_-^m \otimes E)_x$.

If $m \geqslant 1$ and $k = 1$, then by counting dimensions we see that this is a proper subspace. Since it is defined by the *first* order neighbourhood of L_x in Z it is a $CE(4) \times G$-invariant submodule and so by the conformal invariance arguments of § 1, as x varies $T\alpha$ satisfies the equation $\mathrm{D}_m T\alpha = 0$.

When $m = 0$, p_1^* gives an isomorphism so $T\alpha$ satisfies no first order equation (as we know already by conformal invariance from example 1). Passing to the 2nd order neighbourhood, however, we do obtain a proper subspace:

$$\begin{array}{ccccccc}
0 \to S_+^2 \otimes S_-^2 \otimes E & \to & H^1(\mathcal{O}_Z^2 F(-2)) & \to & H^1(\mathcal{O}_Z^1 F(-2)) & \to & 0 \\
\downarrow & & \downarrow & & \wr\Vert & & \\
0 \to S^2 T^* \otimes E & \to & J_2(E) & \to & J_1(E) & \to & 0.
\end{array}$$

Thus $T\alpha$ satisfies a second order equation $\mathrm{D}_* T\alpha = 0$. We cannot apply conformal invariance to determine D_* here, since second derivatives and curvature intervene; however, invariance does determine the highest order part, or symbol, of D_* which is the same as for D_0: contraction with the symmetric tensor of weight 0 defining the conformal structure. But now D_0 has another property, from § 1: it annihilates the (symplectic) inner product of ϕ and ψ where $\overline{\mathrm{D}}_1 \phi = 0$ and $\mathrm{D}_1 \psi = 0$. We also know now that ϕ corresponds to an element of $H^0(Z, \mathcal{O}(1))$ and a class in $H^1(Z, \mathcal{O}F(-3))$ defines a solution of $\mathrm{D}_1 \psi = 0$. Moreover, the contraction of ϕ and ψ corresponds to the natural product map

$$H^0(Z, \mathcal{O}(1)) \otimes H^1(Z, \mathcal{O}F(-3)) \to H^1(Z, \mathcal{O}F(-2))$$

as may easily be checked. Thus both D_* and D_0 annihilate sections corresponding to the image of this map.

Now unfortunately this image may be zero since $\bar{D}_1\phi = 0$ may have no solutions; however, the property of D_* and D_0 we are using is just a formal property of the 2-jets of ϕ and ψ and we may say that the homomorphisms $J_2(E) \to E$ corresponding to D_* and D_0 both annihilate the image of $H^0(\mathcal{O}^2(1)) \otimes H^1(\mathcal{O}^2F(-3))$ in $J_2(E)_x$. Fortunately every holomorphic section of H on the first order neighbourhood of $L_x \subset Z$ extends uniquely to the 2nd-order neighbourhood:

$$\begin{array}{ccccccc} H^0(\mathcal{O}S^2N^*(1)) & \to & H^0(\mathcal{O}^2(1)) & \to & H^0(\mathcal{O}^1(1)) & \to & H^1(\mathcal{O}S^2N^*(1)) \\ \| \wr & & & & & & \| \wr \\ S^2_+ \otimes H^0(\mathcal{O}(-1)) = 0 & & & & & & S^2_+ \otimes H^1(\mathcal{O}(-1)) = 0. \end{array}$$

Thus, since D_* and D_0 have the same symbol, $D_* - D_0$ is a first order operator whose homomorphism annihilates the image of $H^0(\mathcal{O}^1(1)) \otimes H^1(\mathcal{O}^1F(-3))$ in $J_1(E)_x$. However, this is a conformally invariant subspace (depends on 1-jets) and hence must be the whole of $J_1(E)_x$ (cf. example 1), so $D_* = D_0$.

Hence for all $m \geqslant 0$, $D_m T\alpha = 0$.

Finally suppose we have any solution of $D_m\phi = 0$ on X. We must show that it is of the form $T\alpha$ for some $\alpha \in H^1(Z, \mathcal{O}F(-m-2))$. Since D_m is an elliptic operator with analytic coefficients, ϕ is real analytic: defined by a convergent power series at each point $x \in X$. Now a dimension count shows that the image of $H^1(\mathcal{O}^k F(-m-2))$ in $J_k(S^m_- \otimes E)_x$ satisfies no more linear relations than are generated by the k-jets of solutions of $D_m\phi = 0$, so for each k we may pull back $j_k(\phi)_x$ to an element of $H^1(\mathcal{O}^k F(-m-2))$. Moreover, if B is a sufficiently small metric ball in X centred on x, $p_2^{-1}(B)$ is holomorphically 1-convex (Griffiths 1966), as can be seen by using the distance function, and so the 2nd cohomology of any coherent sheaf vanishes. It follows that the map $H^1(p_2^{-1}(B), \mathcal{O}F(-m-2)) \to H^1(\mathcal{O}^k F(-m-2))$ is surjective and so for each k we can pull back $j_k(\phi)$ to a finite neighbourhood and there exists an element $\alpha_k \in H^1(p_2^{-1}(B), \mathcal{O}F(-m-2))$ such that $T\alpha_k$ agrees with ϕ up to the k-th order of its Taylor expansion at x. As k goes to infinity we obtain a formal solution and it is a question of convergence to find a finite class α such that $T\alpha = \phi$.

This can be achieved by choosing a metric on X and representing α_k by a $(0, 1)$ form which is harmonic in the fibres. The injectivity part of the proof shows that such a form is unique and, since the harmonic forms on P_1 are known explicitly in terms of their cohomology classes, we may estimate the form in terms of its transform $T\alpha_k$. Since ϕ is represented by a convergent power series, we obtain one for α. Injectivity shows that α is unique and hence these local solutions fit together to form a global element in $H^1(Z, \mathcal{O}F(-m-2))$, and the proof of theorem (3.1) is finished.

There is an immediate corollary to the theorem, which when applied to the case $X = S^4$, $Z = P_3(\mathbb{C})$ yields via a theorem of Barth (1978) the complete classification of self-dual connections (Atiyah *et al.* 1978):

Corollary (3.8). *Let X be a compact self-dual space with positive scalar curvature, and E a unitary bundle over X with self-dual connection; then*

$$H^1(Z, \mathcal{O}F(-m-2)) = 0 \quad (m \geqslant 0).$$

Proof. By Theorem (3.1) this space is in 1–1 correspondence with solutions to the field equation $\mathrm{D}_m \phi = 0$. But by the Weitzenböck formula (1.3) or (1.4),

$$0 = \int (\mathrm{D}_m^* \mathrm{D}_m \phi, \phi) = \int (\nabla\phi, \nabla\phi) + \tfrac{1}{12}(m+2)\, R(\phi, \phi) \geqslant 0,$$

with equality iff $\phi = 0$.

Remark. The use of formal neighbourhoods simplifies certain aspects of the transformation between self-dual objects on X and holomorphic ones on Z since it provides a coordinate-free notion of derivative. As an example consider theorem 5.2 of (A.H.S.). This theorem associates to a holomorphic bundle F on Z a bundle E with self-dual connection. In the proof, the reality condition on F is used to define a hermitian structure and the unique (1, 0) connection on F is the pull-back of the required connection on E. The connection can be defined, however, in a complex analytic manner with the aid of the formalism of theorem (3.1).

Recall that the fibre E_x over x is defined as $H^0(Z_x, \mathcal{O}(F))$, or in the notation of Theorem (3.1), $E \cong \mathscr{H}^0(F)$. Now as in (2.5) we have $J_1(E)_x \cong H^0(L_x, \mathcal{O}^1_Y(p_1^* F))$ and so comparing cohomology sequences for the 1st order neighbourhoods of L_x in Z and Y we get the following diagram:

$$\begin{array}{ccccccc}
0 & & & & & & 0 \\
\| & & & & & & \| \\
H^0(N^* \otimes F) & \to & H^0(\mathcal{O}^1_Z F) & \xrightarrow{\cong} & H^0(\mathcal{O}_Z F) & \to & H^1(N^* \otimes F) \\
\downarrow & & \downarrow{\scriptstyle p_1^*} & & \downarrow{\scriptstyle \wr} & & \\
(T^* \otimes E)_x & \to & J_1(E)_x & \to & E_x & &
\end{array}$$

The groups $H^l(N^* \otimes F)$ vanish because F is trivial on the line and $H^l(P_1, \mathcal{O}(-1)) = 0$. Hence the image of p_1^* defines a splitting of the 1-jet sequence: a connection. If F has a real structure then it is clear that the extension of holomorphic sections of F to the first order neighbourhood preserves this and so the connection on E, pulled back to F is the unique $(1,0)$ connection preserving the hermitian structure.

This perhaps helps to explain the footnote on p. 461 of (A.H.S.).

Examples

1. From (3.8) we see that S^4 has no non-trivial solutions to $\mathrm{D}_m \phi = 0$. We may remove a point, however, and obtain $\mathbb{R}^4$ with its standard conformal structure. The twistor space Z of $\mathbb{R}^4$ is the total space of the bundle $H \oplus H \to P_1(\mathbb{C})$ and there is an obvious element of $H^1(Z, \mathcal{O}(-2))$ obtained by pulling back a generator of $H^1(P_1, \mathcal{O}(-2))$. This solution of $\mathrm{D}_0 f = 0$ defines the flat metric on $\mathbb{R}^4$. The pull-backs of classes in $H^1(P_1, \mathcal{O}(-m-2))$ correspond to solutions of $\mathrm{D}_m \phi = 0$ which are

covariant constant relative to the Riemannian connection of the flat metric. A similar result holds for any self-dual space with vanishing Ricci tensor (cf. Penrose 1976) where the twistor-space fibres over $P_1(\mathbb{C})$.

2. This constant solution of $\mathrm{D}_0 f = 0$ in the flat metric may be represented, by applying a conformal inversion of S^4, by $f = 1/r^2$. This, as a scalar of weight 1, is regular at ∞ and has a singularity at the origin. If we take k points $a_i \in \mathbb{R}^4$ and add the corresponding scalars, we get a solution

$$f = 1 + \sum_{i=1}^{k} \frac{m_i^2}{|x-a_i|^2},$$

of $\mathrm{D}_0 f = 0$ on $\mathbb{R}^4 - \bigcup a_i$. This corresponds to an element $f^{\vee}$ of

$$H^1\left(P_3(\mathbb{C}) - \bigcup_{i=0}^{k} L_i,\ \mathcal{O}(-2)\right),$$

where $\{L_i\}$, $0 \leqslant i \leqslant k$ are the lines corresponding to $\{\infty, a_1, \ldots, a_k\} \in S^4$. The function f is used to construct the 't Hooft family of self-dual $SU(2)$ connections on S^4 (A.H.S. § 7). The corresponding holomorphic bundle on $P_3(\mathbb{C})$ may be constructed by using the element $f^{\vee}$ to define an extension of line bundles (Atiyah & Ward 1977).

4. Other cohomology groups

We have seen so far how to interpret the cohomology groups $H^0(Z, \mathcal{O}(n))$ and $H^1(Z, \mathcal{O}F(-m-2))$ for $m \geqslant 0$. It is natural to seek an interpretation also for $H^1(Z, \mathcal{O}F(-m-2))$ for $m < 0$.

Let us consider first the case $m = -1$. Then since $H^1(P_1, \mathcal{O}(-1)) = 0$ restriction to each fibre as in § 3 gives zero. However, this means that we can apply the formulae in the proof of injectivity in theorem (3.1). Namely, we represent $\beta \in H^1(Z, \mathcal{O}F(-1))$ by a $\bar{\partial}$-closed 1-form ω and, since $H^0(P_1, \mathcal{O}(-1)) = 0$ find a unique form $\omega - \bar{\partial} s$ which restricts to zero on each fibre. Pulling back to $S_-^* \backslash 0$ we have a form as in (3.2),

$$\tilde{\omega} = \Sigma s_\alpha \otimes \bar{\sigma}_\alpha^{\vee},$$

and $\bar{\partial}(|\lambda|^2 s_\alpha) = 0$ on each fibre, so since $|\lambda|^2 s_\alpha$ is homogeneous of degree 0, it is constant. Recall now that the indices α refer to a basis of S_+, so we may associate to β at each point a section

$$\phi = \Sigma\, |\lambda|^2 s_\alpha \otimes \psi_\alpha \in \Gamma(S_+ \otimes E).$$

The conformal weight of this section is the sum of two contributions: $\frac{1}{2}$ from $\tilde{\omega}$ and 1 from $|\lambda|^2$ i.e. $\frac{3}{2}$. But now $\tilde{\omega}$ satisfies a differential equation $\bar{\partial}\tilde{\omega} = 0$ and since this depends on just the 1-jet of the conformal structure we obtain by conformal invariance a solution of the Dirac equation $\mathrm{D}_1^* \phi = 0$ (D_1^* is the formal adjoint of D_1: it acts on self-dual spinor fields). Conversely, given ϕ such that $\mathrm{D}_1^* \phi = 0$ it is straightforward to associate to ϕ the corresponding $\tilde{\omega}$ with $\bar{\partial}\tilde{\omega} = 0$. We thus get a one-to-one correspondence between the cohomology group $H^1(Z, \mathcal{O}F(-1))$ and the space of self-dual Dirac fields in a self-dual Yang–Mills background.

For $m < -1$ we may proceed similarly: since $H^1(P_1, \mathcal{O}(|m|-2))$ is zero we again obtain a form $\tilde{\omega} = \Sigma s_\alpha \otimes \bar{\sigma}_\alpha^{\vee}$ where $\bar{\partial}(|\lambda|^2 s_\alpha) = 0$ on each fibre and is homogeneous of degree $|m|-1$ and so we get a section ϕ of $S_+ \otimes S_-^{|m|-1} \otimes E$ which, because $\bar{\partial}\tilde{\omega} = 0$, satisfies a first order equation which we can identify by conformal invariance. The difference in this case is, however, that $\tilde{\omega}$ is not uniquely defined since $H^0(Z_x, \mathcal{O}(|m|-2)) \cong S_-^{|m|-2}$ is non-zero and we may alter the section s by adding any section of $F \otimes H^{|m|-1}$ of the form $\phi^\vee$ (cf. § 2) where $\phi \in \Gamma(S_-^{|m|-2} \otimes E)$. Thus $\tilde{\omega}$ is well defined modulo forms $\bar{\partial}\phi^\vee$. It is now clear what the interpretation is: the cohomology group $H^1(Z, \mathcal{O}F(|m|-2))$ corresponds to the first cohomology of an elliptic complex on X:

$$\Gamma(S_-^{|m|-2} \otimes E) \xrightarrow{d_0} \Gamma(S_+ \otimes S_-^{|m|-1} \otimes E) \xrightarrow{d_1} \Gamma(S_-^{|m|} \otimes E).$$

The operator d_0 is the twistor operator $\bar{D}_{|m|-2}$ extended to the bundle E. It acts on sections of weight $-\frac{1}{2}(|m|-2)$. The operator d_1 is the formal adjoint of the spin $|m|/2$ operator $D_{|m|}$. (Note that any conformally invariant operator $D: \Gamma(E) \to \Gamma(F)$ acting on sections of E of weight w has a conformally invariant adjoint $D^*: \Gamma(F) \to \Gamma(E)$ where E has conformal weight $4-w$.)

We now have the following theorem:

THEOREM (4.1). *Let X be a self-dual space and E a bundle over X with self-dual connection. Then there exists a natural one to one correspondence between the cohomology groups $H^1(Z, \mathcal{O}F(m))$ for $m \geqslant -1$ and:*

(i) *for $m = -1$, solutions to the Dirac equation*

$$D_1^*\phi = 0, \quad D_1^*: \Gamma(S_+ \otimes E) \to \Gamma(S_- \otimes E);$$

(ii) *for $m \geqslant 0$, the first cohomology group of the elliptic complex on X,*

$$\Gamma(S_-^m \otimes E) \xrightarrow{D_m} \Gamma(S_+ \otimes S_-^{m+1} \otimes E) \xrightarrow{D^*_{m+2}} \Gamma(S_-^{m+2} \otimes E),$$

(i.e. the space $\operatorname{Ker} D^*_{m+2}/\operatorname{Im} \bar{D}_m$).

Remarks. 1. When $m = 0$, we obtain part of the de Rham complex tensored with E:

$$A^0(E) \to A^1(E) \to A^2_-(E),$$

where $\bar{D}_0$ is now the covariant derivative of a section of E (E has weight 0). This complex, with E replaced by End E (or more generally the adjoint bundle $\mathfrak{g}$) is important in the study of moduli of self-dual connections (A.H.S., § 6). The 1st cohomology group of the complex gives the infinitesimal deformations of these connections. On the other hand, the cohomology group $H^1(Z, \mathcal{O}\operatorname{End} F)$ measures infinitesimal deformations of the complex structures on F. Hence in this case theorem (4.1) may be regarded as an infinitesimal version of theorem (5.2) of (A.H.S., § 5) whereby there is a global correspondence between self-dual connections on X and holomorphic bundles on Z, trivial on the real lines.

2. The 2nd cohomology of the elliptic complex above, i.e. the cokernel of D^*_{m+2} is, on a compact manifold, naturally dual to the kernel of D_{m+2}, which corresponds

from theorem (3.1) to the group $H^1(Z,\mathcal{O}F(-m-4))$. This duality is essentially Serre duality: the non-degenerate pairing:

$$H^1(Z,\mathcal{O}F(-m-4)) \otimes H^2(Z,\mathcal{O}F^*(m)) \to \mathbb{C},$$

(the canonical bundle K of Z is always isomorphic to H^{-4}). The vanishing theorem of corollary (3.8) may now be used to prove the vanishing of H^2 of the complex and under certain circumstances the Atiyah–Singer index theorem will determine the dimension of H^1, as in the moduli case.

3. The relation between $H^1(Z,\mathcal{O}F(-1))$ and solutions of the self-dual Dirac equation may also be seen by using formal neighbourhoods. We know that simply restricting $H^1(Z,\mathcal{O}F(-1))$ to a line gives zero, but we may also restrict to the first order neighbourhood $H^1(\mathcal{O}_Z^1 F(-1))$. We take the exact cohomology sequence:

$$\begin{array}{ccccccc} H^0(\mathcal{O}F(-1)) & \to & H^1(N^*\otimes F(-1)) & \to & H^1(\mathcal{O}_Z^1 F(-1)) & \to & 0, \\ \| & & \wr\| & & & & \\ 0 & & (S_+\otimes E)_x & & & & \end{array}$$

and we obtain a section ϕ of $S_+\otimes E$. Passing to the second order neighbourhood we can see using the methods of theorem (3.1) that ϕ satisfies the self-dual Dirac equation.

We may try to do the same thing for $m = 0$ to obtain solutions to the self-dual Maxwell equations, but a problem arises. We restrict to the 2nd order neighbourhood since the first two vanish and obtain the exact cohomology sequence:

$$\begin{array}{ccccccc} H^0(\mathcal{O}_Z^2 F) \to & H^0(\mathcal{O}_Z^1 F) & \xrightarrow{\delta} & H^1(S^2N^*\otimes F) & \to H^1(\mathcal{O}_Z^2 F) \to 0 \\ & \wr\| & & \wr\| & \\ & E_x & \xrightarrow{\delta} & (S_+^2\otimes E)_x. & \end{array}$$

(We have seen that $H^0(\mathcal{O}_Z^1 F) \cong H^0(\mathcal{O}_Z F) \cong E_x$, this is just the definition of the connection on E.) The homomorphism δ defines the obstruction to extending a section on the 1st order neighbourhood to the 2nd order neighbourhood of a line, and this is just the *curvature* of the connection on E, $F \in \Gamma(\Lambda_+^2 \otimes \operatorname{End} E)$. Hence $\delta(e) = F(e)$ and $H^1(\mathcal{O}_Z^2 F) \cong (\Lambda_+^2\otimes E)_x/\operatorname{Im} F_x$ and we do not have a well-defined Maxwell field. If the curvature vanishes (and so we effectively have no Yang–Mills field) we do however obtain a self-dual Maxwell field. It is easy to see what this field is, for from theorem (4.1) the group $H^1(Z,\mathcal{O})$ corresponds to the first cohomology H^1_- of the truncated de Rham complex:

$$A^0 \xrightarrow{d} A^1 \xrightarrow{d_-} A^2_-.$$

The corresponding space of Maxwell fields is then just $d_+H^1_- \subset A^2_+$.

This correspondence is, however, globally neither injective nor surjective because on the one hand the cohomology class of a flat line bundle in H^1_- defines a zero Maxwell field, and on the other all 2-forms in $d_+H^1_-$ are cohomologically trivial.

We might further attempt, even locally, to associate a self-dual spin $\frac{3}{2}$ field to an element of $H^1(Z, \mathcal{O}(1))$, but there is again an obstruction unless the space X is conformally flat. If we restrict to 3rd order neighbourhoods and take the exact cohomology sequence we find

$$H^0(\mathcal{O}^3_Z(1)) \to H^0(\mathcal{O}^2_Z(1)) \to \underset{\substack{\| \wr \\ (S^3_+)_x.}}{H^1(S^3N^*(1))} \to H^1(\mathcal{O}^3_Z(1)) \to 0$$

As we saw in the proof of theorem (4.1) every section of H on the first order neighbourhood of a line extends uniquely to the second order neighbourhood. The space $H^0(\mathcal{O}^1_Z(1))$ is an extension $(S_+)_x \to J \to (S_-)_x$. The homomorphism δ measures the obstruction to lifting to the 3rd order neighbourhood and its restriction to $(S_+)_x$ is essentially the self-dual part W_+ of the Weyl tensor, so $H^1(\mathcal{O}^3_Z(1)) \cong (S^3_+)_x$ only if $W_+ = 0$ and the manifold X is conformally flat. Thus only in the flat situation can we obtain self-dual fields in this manner.

4. In the construction of all self-dual solutions to the Yang–Mills equations on S^4, the vector space V which is the kernel of the map

$$A: H^1(\mathcal{O}F(-1)) \otimes H^0(\mathcal{O}(1)) \to H^1(\mathcal{O}F)$$

plays a fundamental role. Theorem (4.1) allows us to give one (admittedly not very physical) interpretation of this space in terms of the manifold S^4. We see, using our theorems, that an element $\alpha = \Sigma\beta_i \otimes s_i$ in the tensor product defines a section $T\alpha = \Sigma T\beta_i \otimes Ts_i \in \Gamma(S_+ \otimes E \otimes S_-)$ of weight $\frac{3}{2} - \frac{1}{2} = 1$, where $T\beta_i$ satisfies the self-dual Dirac equation $\mathrm{D}^*_1\phi = 0$ and Ts_i satisfies the twistor equation $\bar{\mathrm{D}}_1\psi = 0$. It follows by conformal invariance that $T\alpha \in \Gamma(\Lambda^1 \otimes E)$ satisfies $\mathrm{D}^*_2 T\alpha = 0$ and the cohomology class in the complex

$$A^0(E) \to A^1(E) \to A^2_-(E)$$

represented by $T\alpha$ corresponds to $A\alpha \in H^1(\mathcal{O}F)$. But this is zero if $\alpha \in V$, so $T\alpha = \nabla e$ for some section $e \in \Gamma(E)$.

The space V thus represents a distinguished family of sections of the bundle E, whose covariant derivatives may be expressed as products of twistors and self-dual Dirac fields.

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