## DECEMBER 4, 2022

Today we'll examine more properties of the integral.

- Suppose that $f$ is an *even* function, meaning $f(-x) = f(x)$. Then for any $a$ one has

$$\int_{-a}^{a} f(x)\mathrm{d}x = 2\int_{0}^{a} f(x)\mathrm{d}x. \tag{193}$$

- Suppose that $g$ is an *odd* function, meaning $g(-x) = -g(x)$. Then

$$\int_{-a}^{a} g(x)\mathrm{d}x = 0. \tag{194}$$

*Example* 2.84. What is the value of

$$\int_{-5}^{5} \frac{x^3}{x^6+1} = \quad ? \tag{195}$$

The *average value of a continuous function* $f$ defined on a domain $[a, b]$ is

(196) $$\overline{f} = \frac{1}{b-a} \int_a^b f(x)\mathrm{d}x.$$

*Example* 2.85. Think about why this is the matches with your intuition of the average value of a linear function, like $f(x) = 2x$ on the interval $[1, 4]$.

*Example* 2.86. Compute the average value of the function $f(x) = x^2$ on the interval $[1, 4]$.

*Example* 2.87. Compute the average value of the function $f(x) = \sin x$ on the interval $[0, 2\pi]$.

**Proposition 2.88** (Mean value theorem for integrals)**.** *Suppose that $f$ is a continuous function on $[a, b]$. Then, there exists a $c$ with $a < c < b$ such that*

(197) $$f(c) = \overline{f} = \frac{1}{b-a} \int_a^b f(x)\mathrm{d}x.$$

*Proof.* Let $N(x) = \int_a^x f(t)\mathrm{d}t$ be the net area function. By the fundamental theorem of calculus this function is differentiable. Therefore, by the mean value theorem, there exists a $c$ such that

(198) $$N'(c) = \frac{N(b) - N(a)}{b-a}.$$

By the fundamental theorem, the left hand side is exactly $f(c)$. The result follows. □

*Example* 2.89. Find the value of $c$ for the function $f(x) = x^2$ on $[1, 4]$. That is, for which value of $c$ is it true $f(c) = \overline{f}$?

Last time we did an example related to substitution. Let's recall it.

Suppose that $F(x) = \int f(x)\mathrm{d}x$ is an antiderivative of a function $f$. We can use this information to guarantee what the antiderivative of the function $g(x) = xf(x^2)$ is. Indeed, consider the function $G(x) = \frac{1}{2}F(x^2)$. Then, by chain rule

$$G'(x) = \frac{1}{2}F'(x^2) \cdot 2x = xf(x^2). \tag{199}$$

Thus $G(x) = F(x^2)$ is an antiderivative for $g(x) = xf(x^2)$.

Another way to state this computation is

$$\int f(x^2)x\mathrm{d}x = \frac{1}{2}\int f(x^2)\mathrm{d}\left(x^2\right) = \frac{1}{2}\int f(u)\mathrm{d}u \tag{200}$$

where we are using the notation $u(x) = x^2$ and

$$\mathrm{d}u = \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x. \tag{201}$$

This 'substitution rule' is like the anti derivative version of the chain rule. If $u = u(x)$ is a general function of a variable $x$ then we heuristically write

$$\mathrm{d}u = \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x. \tag{202}$$

If $f = f(u)$ is a function of $u$ then obtain

$$f(u)\mathrm{d}u = f(u(x))\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x. \tag{203}$$

On the left hand side we think of everything as a function of the variable $u$, whereas eon the right hand side we think of everything as a function of $x$. This formula leads to the integral equation

$$\int f(u)\mathrm{d}u = \int f(u(x))\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x. \tag{204}$$

Here are the steps to perform substitution to evaluate $\int g(x)\mathrm{d}x$.

- Locate a function $u = u(x)$ whose derivative $\frac{\mathrm{d}u}{\mathrm{d}x}$ is easy to compute which satisfies

$$g(x) = f(u(x))\frac{\mathrm{d}u}{\mathrm{d}x} \tag{205}$$

  for some function $f = f(u)$.
- Use the substitution rule to express $\int g(x)\mathrm{d}x$ as

$$\int g(x)\mathrm{d}x = \int f(u)\mathrm{d}u \tag{206}$$

  where $f(u)$ is an easier function to integrate.
- If $\int f(u)\mathrm{d}u = F(u) + C$ is an anti derivative of $f$, then to finish simply substitute $u = u(x)$

$$\int g(x)\mathrm{d}x = F(u(x)) + C. \tag{207}$$

*Example* 2.90. Find

$$\int \frac{x}{1+x^2}\mathrm{d}x.$$

*Example* 2.91. Find

$$\int x^3 \sin(x^4)\mathrm{d}x.$$