Given a differentiable function $g(x)$ we know we can consider its derivative $g'(x)$.

- Question: Given a function $f(x)$, does there exist a function differentiable function $F(x)$ such that

$$F'(x) = f(x) \quad ? \tag{138}$$

If we can find such a function $F(x)$ we call it an *antiderivative* of $f(x)$.

Example 2.57. What is an antiderivative of the function $f(x) = 1$? How about the function $f(x) = x$? In general, what is an antiderivative of $f(x) = x^n$?

- Question: How unique is an antiderivative? That is, if $F(x), G(x)$ are two antiderivatives of $f(x)$, then how close are $F(x)$ and $G(x)$ to being the same?

Theorem 2.58. *Suppose that $F(x), G(x)$ are two antiderivatives of $f(x)$. Then there exists a constant C such that*

$$F(x) = G(x) + C. \tag{139}$$

Proof. We showed in the Mean Value Theorem section that if $f'(x) = g'(x)$ then $f(x) = g(x) + C$ for some constant C. Apply this to $f = F$ and $g = G$. ☐

Example 2.59. Find all antiderivatives of $\frac{1}{x}$. Next, find all antiderivatives of $\cos(2x)$.

Example 2.60. Are the following statements true?

$$\left(\sin^2 x\right)' = 2 \sin x \cos x$$
$$\left(-\cos^2 x\right)' = 2 \sin x \cos x?$$

If so, what does this say about the antiderivatives of $2 \sin x \cos x$?

We will introduce the following symbol called the integral. We call

$$\int f(x)\mathrm{d}x \tag{140}$$

the *indefinite integral* of $f(x)$. It is meant to represent all possible antiderivatives of $f(x)$. Thus, for example

$$\int x\mathrm{d}x = \frac{1}{2}x^2 + C \tag{141}$$

where C is an arbitrary constant.

- When we say 'evaluate' $\int f(x)\mathrm{d}x$, we mean to find all possible antiderivatives of the function $f(x)$.

Example 2.61. Evaluate $\int \frac{\mathrm{d}x}{1+x^2}$. How about $\int \frac{\mathrm{d}x}{2+x^2}$?

A (first-order) *differential equation* is an equation of the form

$$f'(x) = g(x) \tag{142}$$

or if $y = f(x)$ this reads

$$y' = g(x). \tag{143}$$

A typical problem is to find solutions $y = f(x)$ to this equation. As we just saw this is no uniquely determined. But, if we impose some extra conditions then we can solve for $y = f(x)$ uniquely.

Example 2.62. Find the function $y = f(x)$ defined for $x > 0$ which solves the following equation

$$y' = 3x^2 + \sqrt{x} \tag{144}$$

and satisfies $f(1) = 0$.

Such a problem is called an *initial value problem*.