We have introduced integration as the formal inverse to the derivative. What is the geometric meaning of integration?

Let's take an example. Suppose that the velocity of a car obeys

$$v(t) = 35 \quad \text{miles per hour.} \tag{145}$$

To compute how far the car travels in 2 hours, we simply multiply

$$35\,\frac{\text{mi}}{\text{hr}} \times 2\text{ hr} = 70\text{ mi.} \tag{146}$$

Graphically, $v(t)$ is just a horizontal line. Geometrically, the quantity we just computed = distance traveled, represents the area of the rectangle of height 35 and width 2.

More generally, if $v(t)$ was some interesting function of $t$ we arrive at the following hope / conjecture.

**"Conjecture" 2.63.** *The total distance traveled from time $t_0$ to time $t_1$ by a particle moving with velocity function $v(t)$ is represented by the area under the curve of $v(t)$ on the interval $[t_0, t_1]$.*

To spoil the punchline, the area under the curve will be closely related to integration. For now, we will restrain ourselves to approximating areas under curves using rectangles. In some cases though, we can compute the area directly.

*Example* 2.64. Consider the velocity function $v(t) = t$. How far does the particle travel from time $t = 0$ to time $t = 2$? (Hint: draw a picture.)

Let's consider a slightly more interesting example where a particle travels according the velocity function $v(t) = t^2$. We will approximate how far the particle travels from time $t = 0$ to time $t = 2$ using the following steps.

(1) The first step is to divide the interval $[0, 2]$ into smaller intervals. So, for example, we can divide it into four smaller intervals

$$[0,2] = [0,1/2] \cup [1/2,1] \cup [1,3/2] \cup [3/2,2] \tag{147}$$

(We will always assume that the sub-intervals are equal size).

(2) We then draw rectangles corresponding to each sub-interval whose top left corner touch the graph. So here we would draw four rectangles of width $1/2$ and of heights $v(0) = 0, v(1/2) = 1/4, v(1) = 1, v(3/2) = 9/4$, respectively.

(3) To approximate the area we simply add up the areas of all rectangles. In this case

$$\text{Area} \approx \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 9/4 = \frac{7}{4} = 1.75. \tag{148}$$

The technical term for what we just computed was the **left Riemmann sum approximation** to the area under the curve $v(t) = t^2$ between $t = 0$ and $t = 2$ using four subintervals.

*Example* 2.65. Approximate the same area using eight subintervals.

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$$\text{Area} \approx \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot \frac{1}{16} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{9}{16} + \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot \frac{25}{16} + \frac{1}{4} \cdot \frac{9}{4} + \frac{1}{4} \cdot \frac{49}{16} = \frac{425}{192} \approx 2.21$$

The actual answer is Area = $8/3 \approx 2.67$. We're getting closer!