Last time we introduced the concept of *net area*. Let's recall the precise definition.

Definition 2.67. Let f be a continuous function and consider the region which is bounded by the x-axis and the curve of the graph $y = f(x)$ between $x = a$ and $x = b$. The *net area* is the area of the region above the x-axis minus the area below the x-axis.

Example 2.68. Consider the function $f(x) = 2x - 2$. What is the net area determined by $y = f(x)$ between $x = 0$ and $x = 2$? What about the net area between $x = 0$ and $x = 3$? (Use geometry to obtain the *exact* answer, do not use a Riemann sum approximation.)

Definition 2.69. Consider a function f defined on an interval $[a, b]$ The *definite integral* (if it exists)

$$\int_a^b f(x)\mathrm{d}x \tag{154}$$

is the net area determined by $y = f(x)$ between $x = a$ and $x = b$.

Remark 2.70. This is actually more of a theorem than a definition, but the definition of the definite integral is a little more involved than what we will be focusing on in this class. We'll give a more precise definition in terms of Riemann sums at the end of this lecture.

The existence is guaranteed in two situations.

- the function f is continuous on the interval $[a, b]$, or
- the function f is bounded with only finitely many discontinuities.

Example 2.71. Consider the function $f(x) = \sqrt{9 - x^2}$. Use geometry to compute

$$\int_0^3 \sqrt{9 - x^2}\,\mathrm{d}x. \tag{155}$$

Example 2.72. Using geometry evaluate

$$\int_{-2}^{4} \sqrt{8+2x-x^2}\,\mathrm{d}x. \tag{156}$$

Example 2.73. This example is about definite integrals of even and odd functions.

- Suppose that $f(x)$ is an *even* function and $\int_0^2 f(x)\mathrm{d}x = 7$. Evaluate

$$\int_{-2}^{2} f(x)\,\mathrm{d}x. \tag{157}$$

- Suppose that $g(x)$ is an *odd* function. Evaluate

$$\int_{-2}^{2} g(x)\,\mathrm{d}x. \tag{158}$$

In general, what can you infer about integrals of even/odd functions around intervals that are symmetric about the y-axis?

Let's turn to a precise definition of the definite integral which uses Riemann sums. Recall that the Riemann sum approximating the net area of a function f between $x = a$ and $x = b$ can be written as

$$f(x_1^*) \cdot \Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x \tag{159}$$

where

- n is the number of rectangles/subintervals.
- $\Delta x = (b - a)/n$.
- x_k^* is either the left/midpoint/right point of the kth subinterval depending on the type of Riemann sum we use.

We can write this in a compact form using "sigma-notation"

$$\sum_{k=1}^{n} f(x_k^*)\Delta x. \tag{160}$$

More generally sigma-notation is simply shorthand for expressing a sum of numbers

$$\sum_{k=1}^{n} g(k) = g(1) + g(2) + \cdots + g(n-1) + g(n). \tag{161}$$

There are n terms on the right hand side.

Example 2.74. Evaluate the sum $\sum_{k=1}^{3} k^2$.

More generally, there is a formula for this sum

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \tag{162}$$

(If you like you can check this for small values of n.)

We can now make this idea precise that Riemann sums approximate areas.

Definition 2.75. The definite integral of f from $x = a$ to $x = b$ (if it exists) is the limit of a Riemann sum approximation as the number of subintervals n approaches ∞. That is

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x. \tag{163}$$

Example 2.76. Let's evaluate $\int_1^2 x^2 \mathrm{d}x$ using the definition.

We will use the left Riemann sum approximation. For instance, when $n = 4$ we have $\Delta x = 1/4$ and the left Riemann sum is

$$1^2 \cdot \frac{1}{4} + \left(\frac{5}{4}\right)^2 \cdot \frac{1}{4} + \left(\frac{3}{2}\right)^2 \cdot \frac{1}{4} + \left(\frac{7}{4}\right)^2 \cdot \frac{1}{4}. \tag{164}$$

For general n we have the Riemann sum approximation

$$\begin{aligned}\sum_{k=1}^{n} \left(1 + \frac{k}{n}\right)^2 \frac{1}{n} &= \sum_{k=1}^{n} \frac{(n+k)^2}{n^3} \\ &= \sum_{k=1}^{n} \left(\frac{1}{n} + \frac{2k}{n^2} + \frac{k^2}{n^3}\right) \\ &= 1 + \frac{n+1}{n} + \frac{n(n+1)(2n+1)}{6n^3}.\end{aligned}$$

Taking $n \to \infty$ we obtain

$$1 + 1 + \frac{1}{3} = \frac{7}{3}. \tag{165}$$