If we zoom in, the line tangent to a graph of a smooth function closely approximates the original function. Assume that f is smooth on an interval containing $x = a$. The slope of the line tangent to the curve $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$. The equation of a line tangent to the graph at this point, in slope intercept form, is

$$y - f(a) = f'(a)(x - a) \tag{113}$$

or

$$y = f(a) + f'(a)(x - a). \tag{114}$$

Let

$$L_a(x) = f(a) + f'(a)(x - a) \tag{115}$$

be the function describing this line. Then, we say that that $L_a(x)$ is the linear approximation of $f(x)$ near $x = a$ and will often write

$$f(x) \approx L(x) \tag{116}$$

for x near a.

Example 2.47. Find the linear approximation for the function $f(x) = \ln x$ and use it to approximate $\ln(1.1)$.

Example 2.48. Find the linear approximation of $f(x) = \tan^3(x)$ and use it to approximate $\tan(11\pi/40)$. (Notice that $11/40 \approx 1/4$.)

We are trying to approximate the value of the function $f(x) = \tan^3(x)$ near $x = \pi/4$. The derivative is

$$f'(x) = 3\tan^2(x)\sec^2(x). \tag{117}$$

The line $y = L(x)$ tangent to the graph at $x = \pi/4$ is

$$L(x) - f(\pi/4) = f'(\pi/4)(x - \pi/4). \tag{118}$$

Let's record the relevant values of the function and the derivative:

$$f(\pi/4) = 1, \quad f'(\pi/4) = 3 \cdot 1 \cdot (\sqrt{2})^2 = 6. \tag{119}$$

So plugging all of this in:

$$L(x) = 6(x - \pi/4) + 1. \tag{120}$$

Let's move on to L'Hôpital's rule. The most basic application of this rule is to compute limits of the following form

$$\lim_{x \to a} \frac{f(x)}{g(x)} \tag{121}$$

where $f(a) = g(a) = 0$.

If f, g are continuous and differentiable functions near $x = a$, we can use a linear approximation to compute either of the limits $\lim_{x \to a} f(x)$ or $\lim_{x \to a} g(x)$. Take $f(x)$ for example: near $x = a$ we have seen that $f(x)$ is approximated by the linear function

$$L_f(x) = f(a) + f'(a)(x - a). \tag{122}$$

Notice that

$$\lim_{x \to a} f(x) = \lim_{x \to a} L(x) = f(a). \tag{123}$$

Similarly, there is a linear function

$$L_g(x) = g(a) + g'(a)(x - a) \tag{124}$$

which approximates g near $x = a$.

We can attempt to use these approximations to compute the original limit $\lim_{x \to a} \frac{f(x)}{g(x)}$. Indeed, near $x = a$ we have

$$\frac{f(x)}{g(x)} \approx \frac{L_f(x)}{L_g(x)} = \frac{f(a) + f'(a)(x - a)}{g(a) + g'(a)(x - a)}. \tag{125}$$

But the assumption was that $g(a) = f(a) = 0$, so

$$\frac{f(x)}{g(x)} \approx \frac{f'(a)(x - a)}{g'(a)(x - a)}. \tag{126}$$

Notice that for $x \neq a$ we have shown that

$$\frac{f(x)}{g(x)} \approx \frac{f'(a)}{g'(a)} \tag{127}$$

One can turn this into the following theorem.

Theorem 2.49. *Suppose that f, g are differentiable on an interval containing $x = a$ and assume that $g'(x) \neq 0$ on this interval for $x \neq a$. Also, assume that $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$. Then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}. \tag{128}$$