Let's return to a real-world application of derivatives. We've discussed that if $s(t)$ is the position of a particle as a function of time then the derivative $s'(t)$ represents the velocity as a function of time:

$$s'(t) : \quad \text{velocity at time } t. \tag{49}$$

In other words, the rate of change of position is velocity.

As we say last time, we can iterate the process of taking the derivative. In other words, we can consider the second derivative $s''(t)$ This represents the rate of change of velocity as a function of time and is called the *acceleration*.

Example 2.12. Suppose a ball was thrown off of a cliff into the water at time $t = 0$ and its height (in meters) above sea level (before hitting the water) is described by the following function if time (in seconds):

$$s(t) = 10 + 23t - 5t^2. \tag{50}$$

At what time does the ball hit the water? At what time is the ball moving the fastest? Describe the acceleration of the ball. Can you determine the horizontal displacement of the ball?

Chain rule. Now we will move onto the 'chain rule'. First, we need to remind ourselves how to compose functions. Given functions f and g such that the image of g is contained inside the domain of f, the composition $h = f \circ g$ is defined. One also write $h(x) = f(g(x))$.

As an example, consider the functions $f(x) = x + 1$ and $g(x) = \sqrt{x}$. Then

$$f(g(x)) = f(\sqrt{x}) = \sqrt{x} + 1.$$

Notice that the image of g is the non-negative real numbers. Since f is defined everywhere, the function $f \circ g$ is defined wherever g is defined, which is the non-negative real numbers.

In this example we can also consider the composition $g \circ f$ which is

$$g(f(x)) = g(x+1) = \sqrt{x+1}.$$

Note here that the image of f is all real numbers, so in order for the composition to be defined we must restrict f to a smaller domain. In this example, the composition $g \circ f$ is defined whenever $x \geq -1$.

Example 2.13. Write the following functions as compositions of two functions

$$e^{x^2}, \quad \sqrt{\sin(x) + 3}, \quad \sin(x^2). \tag{51}$$

The chain rule is a result which expresses the derivative of a composition of two functions $f \circ g$ in terms of the derivatives of the original two functions f and g.

Theorem 2.14. *Suppose that f, g are differentiable functions and that the composition is well-defined. Then $f \circ g$ is differentiable and*

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x). \tag{52}$$

Sometimes it is useful to introduce more variables. Write $y = f(x)$ so that in this expression x is the independent variable and y is the dependent variable. On the other hand, we write $u = g(y)$ so that in this expression y is the independent variable and u is the dependent variable. Since y depends on x and u depends on x, we see that u depends on x.

Using this notation, we sometimes write derivatives as

$$f' = \frac{\mathrm{d}y}{\mathrm{d}x}, \quad g' = \frac{\mathrm{d}u}{\mathrm{d}y}.$$

Then, heuristically we can view the chain rule as the following formula

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}y}. \tag{53}$$

Example 2.15. Find the derivative of

$$f(x) = \sqrt{e^x + 1}. \tag{54}$$

Example 2.16. Find the derivative of

$$\frac{1}{(2x^2 + 3)^{10}}. \tag{55}$$