Exponentials. Today we continue with rules of derivatives, starting with the exponental function.

Definition 1.4. The exponential function e^x is defined by

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n \tag{18}$$

This number is defined for all real numbers x. Approximately, one has

$$e \stackrel{\text{def}}{=} e^1 = 2.71828\ldots. \tag{19}$$

The exponential function obeys the standard rules of exponentials.[1] For instance, $e^0 = 1$. Another good function to keep in mind is the natural logarithm which is the inverse to the exponential function:

$$\ln(e^x) = e^{\ln x} = x \tag{20}$$

The natural logarithm $\ln x$ is only defined for $x > 0$. A good limit to keep in mind is:

$$\lim_{x\to\infty} e^x = \infty. \tag{21}$$

Let's turn to the derivative of the exponential function. First we have the following lemma.

Lemma 1.5. *One has*

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

Proof. Let's use equation (18) to write

$$\frac{e^h - 1}{h} = \frac{1}{h} \lim_{n\to\infty} \left(-1 + \left(1 + \frac{h}{n}\right)^n\right). \tag{22}$$

Next, we use the binomial theorem to write

$$-1 + \left(1 + \frac{h}{n}\right)^n = -1 + \left(1 + n\frac{h}{n} + \cdots\right) = h + \cdots$$

where the $\cdots$ stands for terms which are at least quadratic in the parameter h. Plugging this back into (22) we obtain

$$\frac{e^h - 1}{h} = 1 + \cdots \tag{23}$$

where now the $\cdots$ stands for terms that are at least linear in h. Taking the limit $h \to 0$ yields the result. □

Example 1.6. Show that if $f(x) = e^x$ is the exponential function then

$$f'(x) = e^x. \tag{24}$$

In other words the exponential is its own derivative $(e^x)' = e^x$. This fact makes it very useful to model population growth, interest, etc. via the exponential function.

[1] Actually proving that $e^x = (e^1)^x$ is a good exercise.

If $f = f(x)$ is a function, define the second derivative to be

$$f''(x) = (f'(x))'. \tag{25}$$

That is, this is the derivative of the derivative. Similarly we can define the third derivative $f'''(x)$, and so on.

Recall that we have introduced the notation

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x}. \tag{26}$$

We will also use

$$f''(x) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \tag{27}$$

and so on.

Example 1.7. Let $f(x) = 3x^3 - 12\sqrt{x}$. Find $f''(x)$.

Example 1.8. Find the twenty-first derivative of the function $f(x) = e^x - 3x^{12}$.

Example 1.9. Compute the limit

$$\lim_{a \to 1} \frac{\sqrt{3 + a} - 2}{a - 1} \tag{28}$$

(Hint: Don't compute the limit.)