

• Composition of functions.

If $f: A \rightarrow B$ is a function between sets,
and so is $g: B \rightarrow C$, then we define the
composition $g \circ f: A \rightarrow C$ by:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ & \searrow & & \nearrow & \\ & g \circ f & & & \end{array} : (g \circ f)(a) = g(f(a))$$

Prop: Suppose $T: V \rightarrow W$ and $S: W \rightarrow Z$ are
linear transformations. Then

$$S \circ T: V \rightarrow Z$$

is linear.

$$\begin{aligned} \text{Pf: } S(T(x + \lambda y)) &= S(T(x) + \lambda T(y)) \\ &= S(T(x)) + \lambda S(T(y)) \\ &= (S \circ T)(x) + \lambda (S \circ T)(y). \end{aligned}$$



Sps that: $T: V \rightarrow W, S: W \rightarrow Z$.

V.S.: $V \quad W \quad Z$

basis: $\alpha = \{v_i\}_{i=1}^n \quad \beta = \{w_j\}_{j=1}^m \quad \gamma = \{z_k\}_{k=1}^p$

$$[T]_{\alpha}^{\beta} = B = (b_i^j) \quad m \times n \text{ matrix.}$$

$$[S]_{\gamma}^{\beta} = A = (a_j^k) \quad p \times m \text{ matrix.}$$

Then:

$$(S \circ T)(v_i) = S\left(\sum_{j=1}^m b_i^j w_j\right)$$

$$= \sum_{j=1}^m b_i^j S(w_j) \quad \text{linearity of } S.$$

$$= \sum_{j=1}^m b_i^j \cdot \sum_{k=1}^p a_j^k z_k$$

$$= \sum_{j=1}^m \sum_{k=1}^p a_j^k b_i^j z_k$$

$$= \sum_{k=1}^p c_i^k z_k$$

where

$$c_i^k = \sum_{j=1}^m a_i^j b_j^k.$$

We can view $C = (c_i^k)$ as a $p \times n$

matrix! We denote it $C \stackrel{\text{def}}{=} AB$.

matrix multiplication

We have proven:

Theorem:

$$[S \circ T]_{\alpha}^{\gamma} = [S]_{\beta}^{\gamma} \cdot [T]_{\alpha}^{\beta}.$$

$\parallel$

C

A

B

$\hat{=}$

matrix mult.



"matrix multiplication is composition"

Summary:

• $A = (a_i^j)$ is $p \times m$ matrix.

• $B = (b_j^k)$ is $m \times n$ matrix.

Then AB is the $p \times n$ matrix with:

$$(AB)_i^k = \sum_{j=1}^m a_i^j b_j^k.$$

Ex: $A = m \times n$ matrix. $v = n \times 1$ matrix.
 $= (a_{ij})$ $= (v_j)$

Av is an $m \times 1$ matrix.

$$Av = \left[\begin{array}{c} \sum_{j=1}^n a_{1j} v_j \\ \sum_{j=1}^n a_{2j} v_j \\ \vdots \\ \sum_{j=1}^n a_{mj} v_j \end{array} \right] \left. \vphantom{\sum_{j=1}^n} \right\} m \text{ rows.}$$

Theorem: If $T: V \rightarrow W$ is linear

and β is basis for V , γ basis for W then:

$$[Tv]_{\gamma} = [T]_{\beta}^{\gamma} \cdot [v]_{\beta}$$

"Snake goes down hole".

This is just a special case of matrix multiplication

• Let $\mathbb{1}_n$ be the $n \times n$ "identity" matrix

$$\mathbb{1}_n = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

Theorem: Let A be $m \times n$ matrix

B, C be $n \times p$ matrices.

D, E be $q \times m$ matrices

Then:

$$1) A(B+C) = AB + AC, (D+E)A = DA + EA.$$

$$2) a(AB) = (aA)B = A(aB) \text{ for any scalar } a.$$

$$3) \mathbb{1}_m A = A \mathbb{1}_n.$$

Pf:

Ex: Let $\int : P_2 \rightarrow P_3$ be

$$\int(f) = \int_0^x f(t) dt.$$

Then $\int$ is linear. And with respect to the standard bases

$$\beta = \{1, x, x^2\}$$

$$\gamma = \{1, x, x^2, x^3\}$$

$$[\int]_{\beta}^{\gamma} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}.$$

If $f(x) = a + bx + cx^2$ then:

$$[f]_{\beta} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Now:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ 1/2 b \\ 1/3 c \end{bmatrix}$$

Exactly the same:

$$\int (a + bx + cx^2) = ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3.$$

- We have seen that given any $T: V \rightarrow W$ and bases that we can form the matrix which "represents" T : $[T]_{\beta}^{\gamma}$.

Conversely: if A is a $m \times n$ matrix, then we can define the linear transformation.

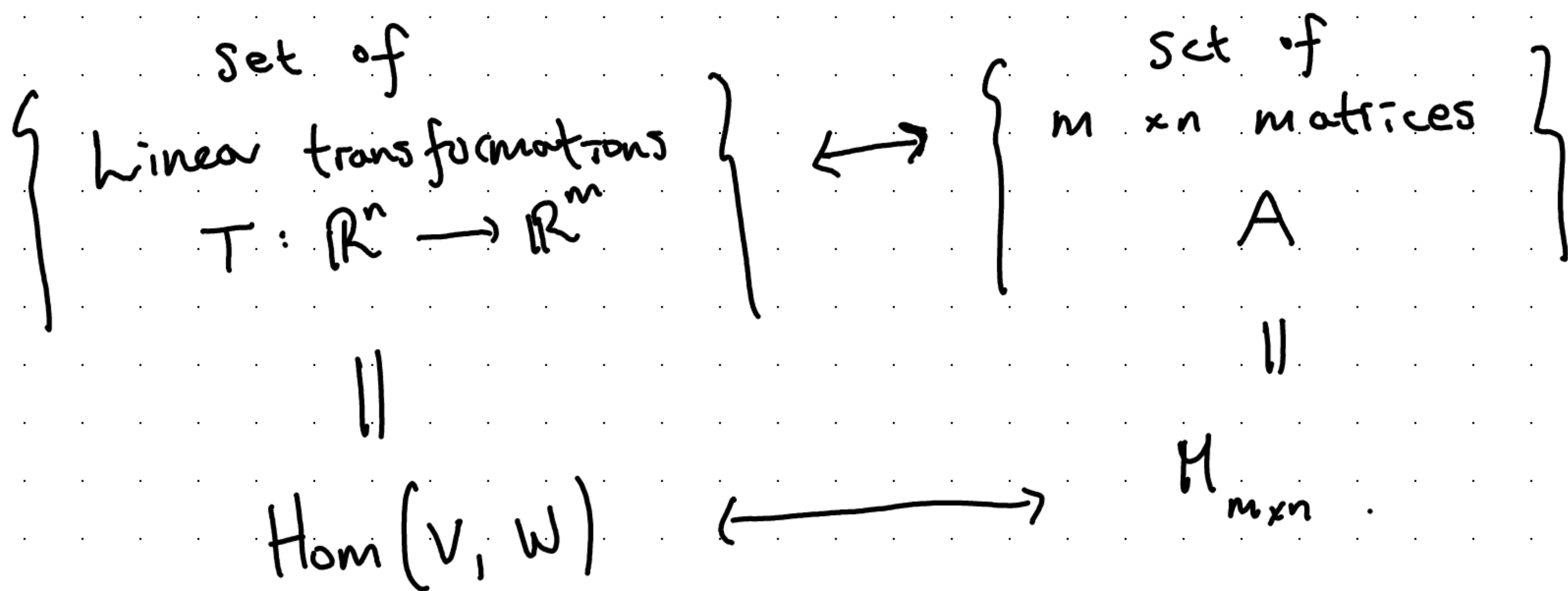
$$L_A: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$v \longmapsto A \cdot v$$

||
matrix multiplication.

Theorem: There is a bijection

(in fact an isomorphism, back to this later).



Explicitly, a matrix A corresponds to the linear

$$\text{map } L_A: \mathbb{R}^n \rightarrow \mathbb{R}^m.$$

Prop: $L_{AB} = L_A \circ L_B$. [we showed this earlier].

We can deduce associativity for matrix multiplication:

Theorem: $A(BC) = (AB)C$.

But AB may not equal BA

Ex: $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
 $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq$