

Invertibility: Let $T: V \rightarrow W$ be a linear transformation.

A function $S: W \rightarrow V$ is an inverse of T

if

$$T \circ S = \mathbb{1}_W: W \rightarrow W$$

$$S \circ T = \mathbb{1}_V: V \rightarrow V.$$

Facts: • S is unique if it exists.

[If S, S' are two inverses, then

$$S = S \circ \mathbb{1}_W \quad (\text{identity})$$

$$= S \circ (T \circ S') \quad (S' \text{ is inverse})$$

$$= (S \circ T) \circ S' \quad (\text{composition is associative})$$

$$= \mathbb{1}_V \circ S' \quad (S \text{ is inverse})$$

$$= S' \quad \Rightarrow \quad S = S'. \quad]$$

We will now denote the inverse of T
by T^{-1} (when it exists).

• If T^{-1} exists, then T is injective.

{ Sps $T(x) = T(y)$. Then

$$T^{-1}(T(x)) = T^{-1}(T(y))$$

$\parallel$

$\parallel$

$$(T^{-1} \circ T)(x)$$

$\vdots$

$\vdots$

$\parallel$

$$1(x)$$

$\parallel$

$\parallel$

$$=$$

y

}

• If T^{-1} exists it is surjective.

{ Sps $y \in W$. Then $T^{-1}(y) \in V$ satisfies

$$T(T^{-1}(y)) = y. \quad \}$$

... So T is bijective if T^{-1} exists.

• If T^{-1} exists, it is linear.

[Sps $y_1, y_2 \in W$. Since T is surjective:

$$y_1 = T(x_1), \quad y_2 = T(x_2)$$

for some x_1, x_2 . Then:

$$T^{-1}(y_1 + \lambda y_2) = T^{-1}(T(x_1) + \lambda T(x_2))$$

$$= T^{-1}(T(x_1 + \lambda x_2))$$

$$= (T^{-1} \circ T)(x_1 + \lambda x_2)$$

$$= x_1 + \lambda x_2$$

$$= T^{-1}(y_1) + \lambda T^{-1}(y_2) -$$

}

We say T is invertible if T^{-1} exists.

Ex: $T: \mathbb{P}_2 \rightarrow \mathbb{R}^3$

$$T(a_0 + a_1x + a_2x^2) = \begin{bmatrix} a_1 \\ a_0 - a_2 \\ a_0 + a_1 \end{bmatrix}$$

Define $S: \mathbb{R}^3 \rightarrow P_2$ by

$$S\left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}\right) = (b_3 - b_1) \cdot 1 + b_1 x + (b_3 - b_1 - b_2) \cdot x^2.$$

Then

$$\begin{aligned} (S \circ T)(a_0 + a_1 x + a_2 x^2) &= S\left(\begin{bmatrix} a_1 \\ a_0 - a_2 \\ a_0 + a_1 \end{bmatrix}\right) = (a_0 + a_1 - a_1) \cdot 1 \\ &\quad + a_1 x + (a_0 + a_1 - a_1 - a_0 + a_2) x^2 \\ &= a_0 + a_1 x + a_2 x^2 \end{aligned}$$

Similarly, $(S \circ T)\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$

• We can also talk about invertible matrices.

Dfn: A square matrix A , size $n \times n$, is invertible

if there exists B s.t.:

1) $AB = I_{n \times n}$

We denote

2) $BA = I_{n \times n}.$

$$B = A^{-1}.$$

Ex: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$ Since:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Dfn: An isomorphism $T: V \rightarrow W$ is an invertible
linear transformation.

Ex: Let $\rho = \{v_i\}$ be a basis for V , of $\dim V = n$.

Then define:

$$T: V \longrightarrow \mathbb{R}^n$$
$$v \longmapsto [v]_{\rho} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

where $v = \sum_{i=1}^n a_i v_i$. Then T is an isomorphism.

It's inverse is: $T^{-1}: \mathbb{R}^n \rightarrow V$ defined by

$$T^{-1}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = \sum_{i=1}^n a_i v_i \in V.$$

Ex: Let $\text{Hom}(V, W) = \left\{ \begin{array}{l} \text{set of linear} \\ \text{maps } V \rightarrow W \end{array} \right\}$.

$$M_{m \times n} = \left\{ \begin{array}{l} \text{set of } m \times n \\ \text{matrices} \end{array} \right\}.$$

Let $\beta = \{v_i\}_{i=1}^n$ basis for V

$\gamma = \{w_j\}_{j=1}^m$ basis for W .

Define $T: \text{Hom}(V, W) \longrightarrow M_{m \times n}$

by the formula

$$T(u) = [u]_{\beta}^{\gamma}.$$

Then T is an isomorphism.

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Theorem: If $T: V \rightarrow W$ is an isomorphism, then

$$\dim V = \dim W.$$

Pf: $\dim V = \dim \ker T + \dim \text{Im } T$

$$= 0 + \dim W = \dim W.$$

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Theorem: Suppose $T: V \rightarrow W$ is linear, and let β basis for V , γ basis for W . If T is invertible then

$$\left[T^{-1} \right]_{\beta}^{\gamma} = \left(\left[T \right]_{\beta}^{\gamma} \right)^{-1}.$$

Pf: As above, $\dim V = \dim W = n$.

$$\mathbb{1}_{n \times n} = \left[\mathbb{1}_V \right]_{\beta} = \left[T^{-1} \cdot T \right]_{\beta}$$

$$= \left[T^{-1} \right]_{\gamma}^{\beta} \cdot \left[T \right]_{\beta}^{\gamma} \quad (\text{Last time}).$$

$$\mathbb{1}_{n \times n} = \left[\mathbb{1}_W \right]_{\gamma} = \left[T \cdot T^{-1} \right]_{\gamma}$$

$$= \left[T \right]_{\beta}^{\gamma} \cdot \left[T^{-1} \right]_{\gamma}^{\beta}$$

This shows: $\left(\left[T \right]_{\beta}^{\gamma} \right)^{-1} = \left[T^{-1} \right]_{\gamma}^{\beta}.$

Theorem: Same setup. If $\left[T\right]_{\beta}^{\gamma}$ is invertible as a matrix then T is invertible as a linear transformation.

Pf: Set $A = (a_j^i)$. Then, A^{-1} exists by assumption.

We need to construct $T^{-1}: W \rightarrow V$ from A^{-1} .

Suppose $A^{-1} = (b_i^j)$.

$$A^{-1}A = \mathbb{I} \Rightarrow \sum_{j=1}^n b_j^i a_k^j = \delta_k^i = \begin{cases} 1, & i=k \\ 0, & i \neq k \end{cases}$$

Then define

$$T^{-1}(w_i) = \sum_{j=1}^n b_i^j v_j$$

Now:

$$(T^{-1} \circ T)(v_i) = T^{-1}\left(\sum_{j=1}^n a_i^j w_j\right)$$

$$= \sum_{j=1}^n a_i^j T^{-1}(w_j)$$

$$= \sum_{j=1}^n \sum_{k=1}^n b_j^k a_i^j v_k$$

$$= v_i \quad \Rightarrow \quad T^{-1} \circ T = \mathbb{I}_V$$

$T \circ T^{-1} = \mathbb{I}_W$ is similar.

