

• Left multiplication: Let $A \in M_{m \times n}$. Define

$$L_A : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

by
$$L_A \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = A \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

↑ Matrix multiplication.

Then L_A is linear.

Dfn: The rank of A is:

$$\text{rank}(A) \stackrel{\text{def}}{=} \dim \text{Im } L_A.$$

Lemma: Suppose $T : V \longrightarrow W$ is invertible. Let

$U : W \longrightarrow Z$, $S : X \longrightarrow V$ be any linear transformations. Then:

1) $\dim \text{Im}(U \circ T) = \dim \text{Im } U.$

2) $\dim \text{Im}(T \circ S) = \dim \text{Im } S.$

3) $\dim \text{Im}(U \circ T \circ S) = \min \{ \dim \text{Im } U, \dim \text{Im } S \}.$

Pf: ① Note that

$$\text{Im } U \circ T \subset \text{Im } U.$$

We will see that the reverse inclusion also holds. Sp. $y \in \text{Im } U$, so $y = U(x)$

for some $x \in W$. Since T is surjective,

$x = T(v)$ for some $v \in V$. Then $y = U(T(v))$

$$\text{so } \text{Im } U \circ T = \text{Im } U.$$

② Similar.

③ Exercise. ~~Q.E.D.~~

Corollary: Let A be an invertible $n \times n$

matrix. Let P be a $p \times n$ matrix and let

Q be a $n \times q$ matrix. Then:

$$\text{① } \text{rank}(PA) = \text{rank}(P).$$

$$\text{② } \text{rank}(AQ) = \text{rank}(Q).$$

$$\text{③ } \text{rank}(PAQ) = \text{rank}(PQ).$$

Elementary row operations.

A $n \times m$ matrix

Three types:

$$\text{So } L_A: \mathbb{R}^m \rightarrow \mathbb{R}^n.$$

1) Swap two rows.

2) Multiply a single row by a nonzero scalar.

3) Add a multiple of one row to a different row.

Eg:

Type 1)

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 7 & 3 & -1 \end{bmatrix} \xrightarrow{-2 \times R_1} \begin{bmatrix} 1 & 0 & 3 \\ -4 & -2 & -2 \\ 7 & 3 & -1 \end{bmatrix}$$

Type 2)

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 7 & 3 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 7 & 3 & -1 \\ 2 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Type 3)

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 7 & 3 & -1 \end{bmatrix} \xrightarrow{R_2 + 2R_1} \begin{bmatrix} 1 & 0 & 3 \\ 4 & 1 & 7 \\ 7 & 3 & -1 \end{bmatrix}$$

Theorem: Let $A \rightsquigarrow \tilde{A}$ be the result of applying an elementary row operation of type 1, 2, 3

Let E be the result of applying exactly the same row operation to $I_{n \times n}$:

$$I_{n \times n} \rightsquigarrow E.$$

Then, $\tilde{A} = E \cdot A$.

Example: Consider the type 2 example above.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\text{Then } E \cdot A = \tilde{A}$$
$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 7 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 3 & -1 \\ 2 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix} \quad \checkmark$$

* Such matrices E , obtained as applying elementary row operation to $I_{n \times n}$ is called an elementary matrix.

Lemma: Elementary matrices are invertible.

Pf: Let E be an elementary matrix of

• type 1: So:

$$\mathbb{I}_{n \times n} \xrightarrow{R_i \leftrightarrow R_j} E$$

Then E^{-1} is:

$$\mathbb{I}_{n \times n} \xrightarrow{R_j \leftrightarrow R_i} E^{-1}$$

• type 2: So $\mathbb{I}_{n \times n} \xrightarrow{\lambda \cdot R_i} E$. Then E^{-1} is:

$$\mathbb{I}_{n \times n} \xrightarrow{\lambda^{-1} \cdot R_i} E^{-1}$$

• type 3: So $\mathbb{I}_{n \times n} \xrightarrow{R_i + \lambda R_j} E$. Then E^{-1} is:

$$\mathbb{I}_{n \times n} \xrightarrow{R_i - \lambda R_j} E^{-1}$$

□

Consequence: If $A \sim \tilde{A}$ is the result of applying elementary row operations then

$$\text{rank } A = \text{rank } \tilde{A}$$

Pf: $\text{rank}(EA) = \text{rank}(A)$ since E is invertible.

□

Theorem: A any matrix. Then

$$\text{rank } A = \# \text{ of linearly independent columns.}$$

Pf: $L_A: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is left multiplication.

By defn, $\text{rank } A = \dim \text{Im } L_A$. Consider the standard basis $\{e_i\}_{i=1}^n$ for $\mathbb{R}^n$. Then we know that:

$$\text{Im } L_A = \text{span} \{L_A(e_1), \dots, L_A(e_n)\}$$

$$= \text{span} \{A \cdot e_1, \dots, A \cdot e_n\}$$

$$= \text{span} \{\text{col } 1, \dots, \text{col } n\}.$$

Since

$$A = [A \cdot e_1 \mid A \cdot e_2 \mid \dots \mid A \cdot e_n].$$

Then, we choose a maximal linearly independent

subset of $\{\text{col } 1, \dots, \text{col } n\}$.

$\square$

Ex: Find the rank of

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

* We can apply row operations until we can easily read off the rank.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{R_3 + R_1} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 10 & 12 \end{bmatrix} \xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 - 4R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + \frac{2}{3}R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{3} \times R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

So, the rank $A = 2$.

Since

$$\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

An algorithm for finding inverse:

Ex: let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Show that A

has rank = 2 (hence is invertible).

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array} \right) \xrightarrow{R_2 - 3R_1} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array} \right)$$

$$\xrightarrow{R_1 + R_2} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array} \right) \xrightarrow{-\frac{1}{2} \times R_2} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array} \right).$$

$\uparrow$
 2×2
This is clearly rank 2.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\Rightarrow \boxed{A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}}.$$

Why did this work?

