

An algorithm for finding inverse:

Ex: let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Show that A

has rank = 2 (hence is invertible).

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array} \right) \xrightarrow{R_2 - 3R_1} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array} \right)$$

$$\xrightarrow{R_1 + R_2} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array} \right) \xrightarrow{-\frac{1}{2} \times R_2} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array} \right).$$

$\uparrow$
 2×2
This is clearly rank 2.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\Rightarrow \boxed{A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}}.$$

Why did this work?

- Why this worked.

Suppose A, B are square matrices $n \times n$. Then an augmented matrix is

$$\underbrace{n \text{ rows } \left\{ \begin{bmatrix} A & | & B \end{bmatrix} \right\}}_{n+n=2n \text{ columns.}}$$

Suppose A is invertible. Then, we can perform row operations

$$A \xrightarrow{E_1} \dots \xrightarrow{E_\ell} \mathbb{I}_{n \times n}.$$

$$\text{That is: } \mathbb{I}_{n \times n} = E_\ell E_{\ell-1} \dots E_2 E_1 \cdot A$$

$$\begin{aligned} \text{Or } A^{-1} &= \left(E_\ell E_{\ell-1} \dots E_1 \right)^{-1} \\ &= E_1^{-1} E_2^{-1} \dots E_{\ell-1}^{-1} E_\ell^{-1}. \end{aligned}$$

But:

$$\begin{aligned} A^{-1} \left[A \mid \mathbb{I}_{n \times n} \right] &= \left[A^{-1} A \mid A^{-1} \cdot \mathbb{I}_{n \times n} \right] \\ &= \left[\mathbb{I}_{n \times n} \mid A^{-1} \right]. \end{aligned}$$

Another example: Determine if

$$A = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 4 & 2 \\ 3 & 3 & 1 \end{bmatrix}$$

is invertible.

If yes, find A^{-1} .

$$\left(\begin{array}{ccc|ccc} 0 & 2 & 4 & 1 & 0 & 0 \\ 2 & 4 & 2 & 0 & 1 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 2 & 4 & 2 & 0 & 1 & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array} \right)$$

$$\xrightarrow{\frac{1}{2} \times R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array} \right)$$

$$\xrightarrow{\begin{array}{l} R_3 - 3R_1 \\ \frac{1}{2} \times R_2 \end{array}} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & -3 & -2 & 0 & -\frac{3}{2} & 1 \end{array} \right)$$

$$\xrightarrow{R_1 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 4 & \frac{3}{2} & -\frac{3}{2} & 1 \end{array} \right)$$

$$\xrightarrow{\frac{1}{4} \times R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{8} & -\frac{3}{8} & \frac{1}{4} \end{array} \right)$$

$$\xrightarrow{\begin{array}{l} R_1 + 3R_3 \\ R_2 - 2R_3 \end{array}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{8} & -\frac{5}{8} & \frac{3}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{8} & -\frac{3}{8} & \frac{1}{4} \end{array} \right)$$

So:

$$A^{-1} = \begin{bmatrix} \frac{1}{8} & -\frac{5}{8} & \frac{3}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ \frac{3}{8} & -\frac{3}{8} & \frac{1}{4} \end{bmatrix}$$

• "Reduced row echelon form" 3.4
= an algorithm for computing ker A.

By $\ker A$, I really mean:

$$\ker L_A = \left\{ \vec{x} \in \mathbb{R}^m \mid A \cdot \vec{x} = \vec{0} \right\}$$

$\swarrow$
 $\ker A$.

Def: A matrix is in reduced row echelon form if:

1) Any row which contains a nonzero entry precedes any row with all 0's.

2) The first nonzero entry in each row is the only nonzero entry in its column.

3) The first nonzero entry in each row is 1 and it occurs to the right of the first nonzero entry of previous row.

Ex: $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is in RREF.

Ex: $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is not in RREF.

• Solving systems of linear equations :

Suppose A is $n \times m$ matrix. Let

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \text{ be } m \times 1 \text{ vector.}$$

$$\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \text{ } n \times 1 \text{ vector.}$$

Computational linear algebra is about finding solutions

$$\vec{x} \text{ to the equation: } A\vec{x} = \vec{b}$$

Three steps : ① First, find all solutions $\vec{x}_0$ to

$$A\vec{x}_0 = \vec{0},$$

the "homogeneous" problem. This is just finding the kernel of A , $\ker A$.

② Second, find a single solution s to $A \cdot s = \vec{b}$.

③ Any other solution to $A\vec{x} = \vec{b}$ is of the form: $\vec{x} = s + \vec{x}_0$, where $\vec{x}_0 \in \ker A$.

$$\underline{\text{Ex}}: \begin{aligned} x_1 + 2x_2 + x_3 &= 7 \\ x_1 - x_2 - x_3 &= -4 \end{aligned}$$

$$\underline{\text{Or}} \quad \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$

$$A \cdot \vec{x} = \vec{b}$$

① Let's put A into RREF by using row operations

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -2 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2/3 \end{bmatrix} \quad \text{This is RREF.}$$

The point is that performing row operations does not change kernel. The new system is much easier:

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ x_2 + \frac{2}{3}x_3 = 0 \end{cases}$$

So: $x_2 = -\frac{2}{3}x_3 = -\frac{2}{3}t$ where $t = x_3$.

and $x_1 = -2x_2 - x_3 = t \frac{4}{3} t - t = \frac{1}{3} t$.

Thus

$$\ker A = \left\{ \begin{bmatrix} \frac{1}{3} t \\ -\frac{2}{3} t \\ t \end{bmatrix} \mid t \in \mathbb{R} \right\}.$$

② Find the particular solution " $\vec{s}$ " to $A \cdot \vec{s} = \vec{b}$.

Lucky guess: $\vec{s} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$.

Check:

$$A \cdot \vec{s} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \end{bmatrix} \checkmark.$$

③ So all solutions are of the form

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + t \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} 1 + t \\ 1 - 2t \\ 4 + 3t \end{bmatrix} \mid t \in \mathbb{R} \right\}.$$