

• Determinants: The determinant of an $n \times n$ matrix

A is an important number:

$$\det A \in \mathbb{R}.$$

• $n=1$, $\det a = a \in \mathbb{R}$.

• $n=2$, $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.

• $n=3$???

Multi-linear functions: Let V, W be vector spaces.

$$V \times V = \{ (v_1, v_2) \mid v_1, v_2 \in V \}$$

this is just a set! Similarly:

$$\underbrace{V \times \dots \times V}_{n\text{-fold}} = \{ (v_1, \dots, v_n) \mid v_i \in V \}.$$

A multi-linear function is a function

$$f : V \times \dots \times V \longrightarrow W$$

satisfying the following:

* For each $v_1, \dots, \hat{v}_j, \dots, v_n \in V$ the function

$$T_{(v_1, \dots, \hat{v}_j, \dots, v_n)} : V \longrightarrow W$$

defined by

$$T_{(v_1, \dots, \hat{v}_j, \dots, v_n)}(v) = \varphi(v_1, \dots, v_{j-1}, v, v_{j+1}, \dots, v_n)$$

is linear. (We call φ an n -linear or multi-linear function).

Ex: $\mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{\times} \mathbb{R}^3$
 $(\vec{v}, \vec{w}) \longmapsto \vec{v} \times \vec{w}$ (cross product).

is a 2-linear function.

Pf: Fix $\vec{v}_0 \in \mathbb{R}^3$. And then

$$T_{\vec{v}_0} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$T_{\vec{v}_0}(\vec{w}) = \vec{v}_0 \times \vec{w} \quad \text{is linear!}$$

Similarly $T_{\vec{w}_0}(\vec{v}) = \vec{v} \times \vec{w}_0$ is a linear function of $\vec{v}$.

for fixed $\vec{w}_0$



The set of all n -linear functions

$\text{Mult}_n(V, W)$ is a vector space.

Lemma: Let β be basis for V . And suppose

$$f: \beta \times \dots \times \beta \longrightarrow W$$

is a function. Then, there exists a unique

n -linear function $\gamma: V \times \dots \times V \longrightarrow W$ such

that

$$\gamma|_{\beta \times \dots \times \beta} = f.$$

The proof is identical to the linear case.

Ex: So, for the cross product you just need

to know $e_i \times e_j$ for $i, j = 1, 2, 3$.

The cross product has another feature:

$$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v} \quad (\text{anti-symmetry}).$$

We will generalize this to n -linear functions.

Dfn: An n -linear function

$$\varphi : V \times \dots \times V \longrightarrow W$$

is alternating if for all pairs $(i < j)$

and vectors $v_1, \dots, v_n \in V$:

$$\varphi(v_1, \dots, v_i, \dots, v_j, \dots, v_n)$$

$$= -\varphi(v_1, \dots, v_j, \dots, v_i, \dots, v_n).$$

[We swapped v_i with v_j .]

So, the cross product is alternating 2-linear function.

Here is another example:

Ex: $\mathbb{R}^2 \times \mathbb{R}^2 \xrightarrow{\det} \mathbb{R}$

$$\begin{bmatrix} a \\ c \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} \longmapsto ad - bc.$$

Check that $\det$ is 2-linear and alternating.

Prop: If $\varphi: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is any alternating 2-linear function, then

$$\varphi = \lambda \det$$

for some $\lambda \in \mathbb{R}$. (That is $\varphi(\vec{v}, \vec{w}) = \lambda \det(\vec{v}, \vec{w})$).

Pf: Consider standard basis. $\{e_1, e_2\}$. Notice:

$$\det(e_1, e_2) = 1 = -\det(e_2, e_1)$$

$$\det(e_1, e_1) = \det(e_2, e_2) = 0.$$

Suppose $\varphi(e_i, e_j) = \varphi_{ij} \in \mathbb{R}$. We know:

$$\begin{cases} \cdot \varphi_{12} = -\varphi_{21} \\ \cdot \varphi_{11} = \varphi_{22} = 0 \end{cases} \quad \text{since } \varphi \text{ is alternating.}$$

Thus $\varphi = \lambda \det$ where $\lambda = \varphi_{12}$. $\square$

• The set of all alternating n -linear fns is subspace:

$$\text{Alt}_n(V, W) \subset \text{Mult}_n(V, W).$$

Theorem: $\text{Alt}_n(\mathbb{R}^n, \mathbb{R}) \cong \mathbb{R}$.

But first:

Lemma: Suppose $\varphi: V \times \dots \times V \rightarrow \mathbb{R}$ is alternating.

Then

$$\varphi(v_1, \dots, v_n) = 0 \quad \text{if } v_i = v_j \text{ for some } i \neq j.$$

Pf. Suppose $v_i = v_j = v$. Then

$$\varphi(\overset{1 \dots i \dots j \dots n}{v_1, \dots, v, \dots, v, \dots, v_n}) = -\varphi(\overset{1 \dots i \dots j \dots n}{v_1, \dots, v, \dots, v, \dots, v_n})$$

Since φ is alternating. □

Pf of theorem: Recall φ is defined by what it does to a basis. Take standard basis of $\mathbb{R}^n$, $\{e_i\}$.

By lemma

$$\varphi(e_{i_1}, \dots, e_{i_n}) = 0 \quad \text{if } i_p = i_q$$

for some $p \neq q$.

So this expression is only nonzero when all of the basis elements are distinct.

Define

$$\lambda = \varphi(e_1, e_2, \dots, e_n).$$

Thus λ completely determines φ by the alternating property. For example:

$$\varphi(e_2, e_1, e_3, \dots, e_n) = -\lambda.$$

In general we can get the value of any

$$\varphi(e_{i_1}, e_{i_2}, \dots, e_{i_n})$$

by doing successive swaps of $\varphi(e_1, e_2, \dots, e_n)$.

Then, define:

$$\text{Alt}_n(\mathbb{R}^n, \mathbb{R}) \xrightarrow{\cong} \mathbb{R}$$

$$\varphi \longmapsto \lambda. \quad \square$$

is the isomorphism.

- We will fix a particularly natural choice of basis for $\text{Alt}_n(\mathbb{R}^n, \mathbb{R})$ called:

$$\det : \mathbb{R}^n \times \dots \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

defined by $\det(e_1, e_2, \dots, e_n) = +1.$

$\square$

• Back to matrices. Define

$$M_{n \times n} \xrightarrow{\text{col}} \mathbb{R}^n \times \dots \times \mathbb{R}^n$$

$$A = (\vec{a}_1 | \dots | \vec{a}_n) \mapsto (\vec{a}_1, \dots, \vec{a}_n)$$

Dfn: The determinant is the composition:

$$M_{n \times n} \xrightarrow{\text{col}} \mathbb{R}^n \times \dots \times \mathbb{R}^n \xrightarrow{\det} \mathbb{R}$$

$$A \mapsto \det A \in \mathbb{R}.$$

$\parallel$ same notation
 $\det(\vec{a}_1, \dots, \vec{a}_n)$

Prop: If $A = (\vec{a}_1 | \dots | \vec{a}_n)$ and

$$\vec{a}_j \in \text{span} \{ \vec{a}_1, \dots, \hat{\vec{a}}_j, \dots, \vec{a}_n \}$$

then $\det A = 0$.

Pf: Sps $\vec{a}_j = \sum_{i \neq j} \lambda_i \vec{a}_i$. Then

$$\det(\vec{a}_1, \dots, \vec{a}_j, \dots, \vec{a}_n) = \det(\vec{a}_1, \dots, \sum_{i \neq j} \lambda_i \vec{a}_i, \dots, \vec{a}_n)$$

$$= \sum_{i \neq j} \det(\vec{a}_1, \dots, \vec{a}_i, \dots, \vec{a}_i, \dots, \vec{a}_n) = 0$$

$\hat{\text{repeated vector!}}$

$\square$

Ex: $\det \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = 0$

Since $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = 2 \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

Ex: $\det \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = ?$

Here is a very important property:

Theorem: $\det(AB) = \det A \cdot \det B$.

Pf: Write $B = [\vec{b}_1 | \dots | \vec{b}_n]$. Then

$$AB = [A\vec{b}_1 | \dots | A\vec{b}_n]$$

Define $y(\vec{b}_1, \dots, \vec{b}_n) = \det(A\vec{b}_1, \dots, A\vec{b}_n)$.

Clearly y is alternating n -linear since $\det$ is both of those things. Thus, by the uniqueness theorem of n -linear alternating fns:

$$y(\vec{b}_1, \dots, \vec{b}_n) = \lambda \det B$$

for some $\lambda \in \mathbb{R}$.

If $(b_1, \dots, b_n) = (e_1, \dots, e_n)$ then

$$\begin{aligned}\varphi(e_1, \dots, e_n) &= \det(Ae_1, \dots, Ae_n) \\ &= \det(\vec{a}_1, \dots, \vec{a}_n) \\ &= \det A\end{aligned}$$

where $\vec{a}_1, \dots, \vec{a}_n$ are the columns of A . Since $\det(e_1, \dots, e_n) = 1$ this shows $\lambda = \det A$. Thus

$$\det(A\vec{b}_1, \dots, A\vec{b}_n) = \det A \cdot \det B$$

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$$\det(AB)$$



Cor: A is invertible $\Leftrightarrow \det A \neq 0$.

Pf $(\Rightarrow) AA^{-1} = \mathbb{1}_{n \times n}$. So:

$$\det A \cdot \det(A^{-1}) = \det \mathbb{1} = 1.$$

So $\det A \neq 0$.

$(\Leftarrow)$ Suppose that $\det A \neq 0$.

Then this means that the columns of

A are all linearly independent. Thus A

is of full rank $\Rightarrow A$ is invertible.

