

Thus λ completely determines φ by the alternating property. For example:

$$\varphi(e_2, e_1, e_3, \dots, e_n) = -\lambda.$$

In general we can get the value of any

$$\varphi(e_{i_1}, e_{i_2}, \dots, e_{i_n})$$

by doing successive swaps of $\varphi(e_1, e_2, \dots, e_n)$.

Then, define:

$$\text{Alt}_n(\mathbb{R}^n, \mathbb{R}) \xrightarrow{\cong} \mathbb{R}$$

$$\varphi \longmapsto \lambda. \quad \square$$

is the isomorphism.

• We will fix a particularly natural choice of basis for $\text{Alt}_n(\mathbb{R}^n, \mathbb{R})$ called:

$$\det : \mathbb{R}^n \times \dots \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

defined by $\det(e_1, e_2, \dots, e_n) = +1.$

$\square$

• Back to matrices. Define

$$M_{n \times n} \xrightarrow{\text{col}} \mathbb{R}^n \times \dots \times \mathbb{R}^n$$

$$A = (\vec{a}_1 | \dots | \vec{a}_n) \mapsto (\vec{a}_1, \dots, \vec{a}_n)$$

Dfn: The determinant is the composition:

$$M_{n \times n} \xrightarrow{\text{col}} \mathbb{R}^n \times \dots \times \mathbb{R}^n \xrightarrow{\det} \mathbb{R}$$

$$A \mapsto \det A \in \mathbb{R}.$$

|| same notation
 $\det(\vec{a}_1, \dots, \vec{a}_n)$.

Prop: If $A = (\vec{a}_1 | \dots | \vec{a}_n)$ and

$$\vec{a}_j \in \text{span} \{ \vec{a}_1, \dots, \hat{\vec{a}}_j, \dots, \vec{a}_n \}.$$

then $\det A = 0$.

Pf: Sps $\vec{a}_j = \sum_{i \neq j} \lambda_i \vec{a}_i$. Then

$$\det(\vec{a}_1, \dots, \vec{a}_j, \dots, \vec{a}_n) = \det(\vec{a}_1, \dots, \sum_{i \neq j} \lambda_i \vec{a}_i, \dots, \vec{a}_n)$$

$$= \sum_{i \neq j} \det(\vec{a}_1, \dots, \vec{a}_i, \dots, \vec{a}_i, \dots, \vec{a}_n) = 0$$

↑ repeated vector!

□

Ex: $\det \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = 0$

Since $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = 2 \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

Ex: $\det \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = ?$

Here is a very important property:

Theorem: $\det(AB) = \det A \cdot \det B$.

Pf: Write $B = [\vec{b}_1 | \dots | \vec{b}_n]$. Then

$$AB = [A\vec{b}_1 | \dots | A\vec{b}_n]$$

Define $y(\vec{b}_1, \dots, \vec{b}_n) = \det(A\vec{b}_1, \dots, A\vec{b}_n)$.

Clearly y is alternating n -linear since $\det$ is both of those things. Thus, by the uniqueness theorem of n -linear alternating fns:

$$y(\vec{b}_1, \dots, \vec{b}_n) = \lambda \det B$$

for some $\lambda \in \mathbb{R}$.

If $(b_1, \dots, b_n) = (e_1, \dots, e_n)$ then

$$\begin{aligned}\varphi(e_1, \dots, e_n) &= \det(Ae_1, \dots, Ae_n) \\ &= \det(\vec{a}_1, \dots, \vec{a}_n) \\ &= \det A\end{aligned}$$

where $\vec{a}_1, \dots, \vec{a}_n$ are the columns of A . Since $\det(e_1, \dots, e_n) = 1$ this shows $\lambda = \det A$. Thus

$$\det(A\vec{b}_1, \dots, A\vec{b}_n) = \det A \cdot \det B$$

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$$\det(AB)$$



Cor: A is invertible $\Leftrightarrow \det A \neq 0$.

Pf $(\Rightarrow) AA^{-1} = \mathbb{1}_{n \times n}$. So:

$$\det A \cdot \det(A^{-1}) = \det \mathbb{1} = 1.$$

So $\det A \neq 0$.

$(\Leftarrow)$ Suppose that $\det A \neq 0$.

Then this means that the columns of A are all linearly independent. Thus A is of full rank $\Rightarrow A$ is invertible. $\square$

• Using row operations to compute determinants.

Lemma: 1) If E is a row operation of type 1
then
 $\det E = -1$.

2) If E is a row operation of type 2:

$$E: \text{row } i \rightsquigarrow \lambda \times \text{row } i$$

then

$$\det E = \lambda$$

3) If E is a row operation of type 3:

$$E: \text{row } i \rightsquigarrow \text{row } i + \lambda \text{ row } j$$

then:

$$\det E = 1$$

Pf : 1) E is obtained from $I_{n \times n}$ by swapping row i with row j . But, this is the same as swapping column i w/ column j since $I_{n \times n}$ is symmetric. So, if $i < j$:

$$I_{n \times n} = [e_1 | \dots | e_i | \dots | e_j | \dots | e_n]$$

↓

$$E = [e_1 | \dots | e_j | \dots | e_i | \dots | e_n]$$

$$\begin{aligned} \text{Thus } \det E &= \det(e_1, \dots, e_j, \dots, e_i, \dots, e_n) \\ &= -\det(e_1, \dots, e_i, \dots, e_j, \dots, e_n) \\ &= -1 \end{aligned}$$

2) Use multi-linearity of $\det$.

$$\begin{aligned} \det E &= \det[e_1 | \dots | \lambda e_i | \dots | e_n] \\ &= \det(e_1, \dots, \lambda e_i, \dots, e_n) \\ &= \lambda \det(e_1, \dots, e_i, \dots, e_n) = \lambda \end{aligned}$$

3 | This time $\bar{E}$ looks like

$$\bar{E} = [e_1 | \dots | \overbrace{e_i}^i + \lambda e_j | \dots | e_n]$$

So:

$$\begin{aligned} \det \bar{E} &= \det(e_1, \dots, e_i + \lambda e_j, \dots, e_n) \\ &= \det(e_1, \dots, \overbrace{e_i}^i, \dots, e_n) + \lambda \det(e_1, \dots, \cancel{e_i}, \dots, \cancel{e_j}, \dots, e_n) \\ &= 1. \end{aligned}$$

$\square$

Ex: Compute $\det \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 4 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 4 \end{bmatrix} \xrightarrow{\text{type 1}} \begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{type 1}} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{\text{type 3}} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow[\lambda = -1/2]{\text{type 2}} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{type 3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det I_{3 \times 3} = (-1)(-1)(-1/2) \cdot \det A \Rightarrow \boxed{\det A = -2}$$

So: we did two type 1 moves $(-1) \times (-1) = 1$
 and one type 2 move $\lambda = -1/2$
 and a bunch of type 3 moves.

So: $\det \mathbb{1}_{n \times n} = 1$

$$\Rightarrow \boxed{\det A = -2.}$$

$$(-1)(-1)\left(\frac{1}{2}\right)(-1) \det A$$

Prop: Suppose

$$A = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \boxed{B} \\ \vdots & & & \\ 0 & & & \end{bmatrix}$$

$A \ n \times n$

$B \ (n-1) \times (n-1).$

Then $\boxed{\det A = \det B.}$

Pf: Let $W \subset \mathbb{R}^n$ be the subspace

$$W = \text{span} \{e_2, e_3, \dots, e_n\} \subset \mathbb{R}^n.$$

$$\cong \downarrow T$$

$$\mathbb{R}^{n-1}$$

where T is the linear isomorphism.

$$T \begin{bmatrix} 0 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{n-1}.$$

Let $\tilde{e}_1, \dots, \tilde{e}_{n-1}$ be the standard basis for $\mathbb{R}^{n-1}$.

$$\tilde{e}_j = T(e_{j+1}).$$

T is invertible w/ inverse

$$T^{-1} \begin{bmatrix} y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}.$$

Let $B = [\vec{b}_1 | \dots | \vec{b}_{n-1}]$, $\vec{b}_j \in \mathbb{R}^{n-1}$.

Then

$$\det A = \det(e_1, T^{-1}(\vec{b}_1), \dots, T^{-1}(\vec{b}_{n-1})).$$

Now: Consider the multilinear function

$$\varphi : \mathbb{R}^{n-1} \times \dots \times \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$$

defined by $\varphi(\vec{y}_1, \dots, \vec{y}_{n-1}) = \det(e_1, T^{-1}(\vec{y}_1), \dots, T^{-1}(\vec{y}_{n-1}))$.

Then φ is alternating since $\det$ is!

Thus there is some $\lambda \in \mathbb{R}$ such that $\varphi = \lambda \det$.

Note, when $y_1 = \tilde{e}_1, \dots, y_{n-1} = \tilde{e}_{n-1}$ then

$$\begin{aligned} \varphi(\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_{n-1}) &= \det(e_1, e_2, e_3, \dots, e_{n-1}, e_n) \\ &= 1 \end{aligned}$$

$$\Rightarrow \lambda = 1 \text{ and } \det A = \det B.$$



Application: a general formula for $\det A$ when

A is 3×3 matrix. (Really, this can be generalized to any dimension.)

Prop:

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_2c_1 - b_1c_2).$$

Pf: Write $\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3$.

By multilinearity (in the first factor):

$$\det(\vec{a}, \vec{b}, \vec{c}) = a_1 \det(\vec{e}_1, \vec{b}, \vec{c}) + a_2 \det(\vec{e}_2, \vec{b}, \vec{c}) + a_3 \det(\vec{e}_3, \vec{b}, \vec{c}).$$

Consider first term:

$$\det \begin{bmatrix} 1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{bmatrix} \stackrel{\text{Last Prop.}}{=} \det \begin{bmatrix} b_2 & c_2 \\ b_3 & c_3 \end{bmatrix} = b_2c_3 - b_3c_2.$$

Second term:

$$\det \begin{bmatrix} 0 & b_1 & c_1 \\ 1 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{bmatrix} \stackrel{\text{row flip}}{=} - \det \begin{bmatrix} 1 & b_2 & c_2 \\ 0 & b_1 & c_1 \\ 0 & b_3 & c_3 \end{bmatrix}$$
$$= - \det \begin{bmatrix} b_1 & c_1 \\ b_3 & c_3 \end{bmatrix} = - (b_1 c_3 - b_3 c_1).$$

Finally.

$$\det \begin{bmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 1 & b_3 & c_3 \end{bmatrix} \stackrel{\text{row flip}}{=} - \det \begin{bmatrix} 1 & b_3 & c_3 \\ 0 & b_2 & c_2 \\ 0 & b_1 & c_1 \end{bmatrix}$$
$$= - \det \begin{bmatrix} b_2 & c_2 \\ b_1 & c_1 \end{bmatrix} = b_1 c_2 - b_2 c_1. \quad \square$$

This formula is called the
"cofactor expansion".

Try proving the general $n \times n$ case.